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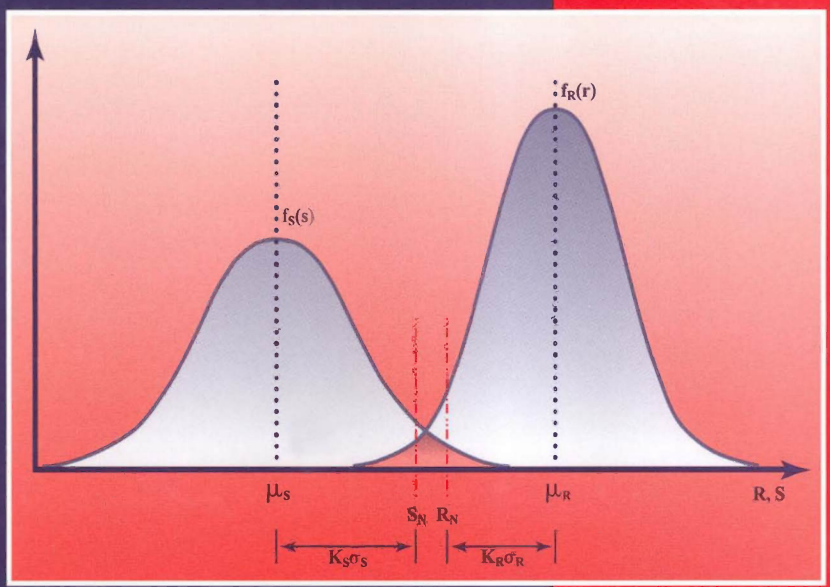
Probability, Reliability and Statistical Methods in Engineering Design



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Dedication

To my wife Carolyn for her help in all phases of the development of this book, our son Justin, and my mother Angur Haldar.

Achintya Haldar

To my parents, Ganesan Sankaran and Janaki Sankaran, and my wife Monica.

Sankaran Mahadevan

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Preface

The application of risk and reliability in the analysis, design, and planning of engineering systems has received worldwide acceptance from the engineering profession. As a result of extensive efforts by different engineering disciplines during the last three decades, design guidelines and codes are being modified or have already been modified to incorporate the concept of risk-based analysis and design. The Accreditation Board of Engineering and Technology (ABET) now requires that all civil engineering undergraduate students in the U.S. demonstrate knowledge of the application of probability and statistics to engineering problems, indicating its importance in civil engineering education.

There are many motivations behind this book, and they need some elaboration here. In the 1970s, while deeply involved with safety issues for nuclear power plants, the first author observed that ordinary engineers were not trained in risk or reliability-based analysis and design procedures. This is true even today to some extent. Many engineers are reluctant to use these procedures because reliability methodology appears to be difficult and mathematically demanding. The available literature is not easy to read, and the basic concept is buried in complex mathematical notations, symbols, and diagrams. Information is available on a piecemeal basis emphasizing a few areas of interest to the writers, and the numerous articles fail to give a broad, comprehensive understanding of the current state of the art. Some of the research issues and items of interest to academicians are emphasized; however, some basic concepts that might facilitate practical understanding and implementation are overlooked or deemphasized.

For engineers who understand risk-based analysis and design, applying the concept to engineering problems still appears to be very difficult. This is because the basic theories and algorithms have been under considerable research, development, and verification efforts until recently. There has been much ferment in this field during the past two decades, resulting in numerous publications with various methods and theoretical arguments. However, much experience has been gained from this research during the past decades, and now it is time to crystallize these concepts into a simple, ready-to-use format to enable risk-based design. This is an important motivation behind this book.

In this book we strive to deal comprehensively with issues relevant to students, professionals, and society. There are several target audiences for this book. The book addresses issues of interest to undergraduate and graduate students of civil engineering, and to other engineering disciplines to a lesser extent; it will also interest practicing engineers with little or no background in probability and statistics, and the general risk and reliability research community. The material is presented attractively and efficiently and is based on a great deal of experience gained by dealing with students and practicing engineers on safety-related issues.

To satisfy the objectives of the book and the target audiences, we develop the material gradually, first addressing very simple concepts and later moving to advanced concepts required to implement risk-based design concepts in practical problems. The statistical knowledge required to implement the concept is presented in an optimal way, fitting it into the overall scheme of risk or reliability evaluation of engineering systems. Many reliability analysis methods with various degrees of complexity and sophistication are presented. The area of simulation is becoming an integral part of risk and reliability analysis, even with an elementary knowledge of probability and statistics. The extensive use of personal computers is the motivation behind the chapter on simulation techniques. Simulation also plays an important role in verifying the results obtained using more mathematically demanding analytical methods.

The material is developed primarily for undergraduate engineering students; however, it can be used by graduate engineering students as well and by practicing engineers with no formal instruction on the subject. The material is presented directly and straightforwardly with many practical examples, in an attempt to increase appreciation for the subject and demystify the complicated mathematical theory of risk and reliability. An understanding of the contents of the book is expected to provide a sufficient working knowledge of risk and reliability to all interested parties. It is expected that these chapters can be taught in one semester and will satisfy the ABET requirements.

The book can be used to teach any required undergraduate course for engineers on probability and statistics. These chapters are developed so as to be understandable by members of any engineering discipline. Chapter 1 presents the basic concepts of risk and reliability. Chapters 2 through 8 start with the fundamentals of probability and statistics and present the concepts of reliability analysis, with applications to simple components and systems. The mathematical aspects of evaluating risk and reliability (i.e., set theory) are discussed in Chapter 2. Chapter 3 discusses methods to quantify randomness in terms of data collection and analysis for one or multiple random variables and correlations among random variables. Risk evaluation procedures using some of the common distributions including extreme value distributions are presented in Chapter 4. Statistical concepts regarding the selection of suitable distributions and the estimation of parameters to uniquely describe a selected distribution are discussed in Chapter 5. Chapter 6 discusses several methods for evaluating uncertainty in the response of engineering systems using information on the random variables involved in the problem. Regression analysis procedures are also presented in this chapter. The concept of reliability analysis and currently available risk evaluation procedures are discussed in Chapter 7. Advanced reliability analysis topics are presented in Chapter 8. The use of simulation in estimating risk is introduced in Chapter 9. These chapters are self-contained and can be used in an undergraduate course to satisfy the ABET requirements. Additional materials required to make the book self-sufficient are provided in six appendixes.

This book discusses all the fundamentals of reliability and statistics required for risk-based engineering analysis and design. It primarily emphasizes reliability analysis procedures when the functional relationship among the load and resistance-related variables (i.e., the performance function or the limit state equations) is available in explicit form. The book briefly addresses reliability evaluation procedures when the performance functions are implicit; we discuss this more completely in another book, *Reliability Assessment Using Stochastic Finite Element Analysis*. published by John Wiley & Sons in 2000.

To improve the readability of the book, citations in the middle of a discussion are avoided. Many people contributed to the development of risk and reliability-based engineering analysis and designs, and an extensive list of references is given at the end of the book to acknowledge their contributions. We have tried our best to make this list as complete as possible. We also would like to thank the many anonymous reviewers for their constructive comments and suggestions. Their help was essential in developing this book.

Mr. Jungwon Huh's help in developing figures, tables, and numerical solutions for many problems given in the book is very much appreciated.

Numerous former and present students and colleagues of ours directly or indirectly contributed to the development of the material presented in this book. We would like to thank Prof. B.M. Ayyub at the University of Maryland, Professor T.A. Cruse at Vanderbilt University, Dr. Hari B. Kanegaonkar, Dr. Yiguang Zhou, Dr. Liwei Gao, Dr. Zhengwei Zhao, Dr. Duan Wang, Prof. Alfredo Reyes Salazar, Mr. Rajasekhar K. Reddy, Mr. Nilesh Shome, Mr. Ali Mehrabian, Mr. Seung Yeol Lee, Dr. Sandeep Mehta, Dr. Robert Tryon, Dr. Qiang Xiao, Dr. Animesh Dey, Dr. Xiaoping Liu, Mr. Pan Shi, and Mr. Zhisong Guo.

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NOTES FOR INSTRUCTORS

The book is suitable for a sophomore or junior level course as well as for a senior or first-year graduate level course. Based on extensive teaching experience, we would like to suggest a tentative course outline for the two courses.

Sophomore or Junior Level Course (Average Duration 15 Weeks)

This outline is applicable when the students have very limited exposure to set theory. For this class, the material in Chapters 1 to 3 needs to be covered in detail and can be covered in 10 lectures. Some of the commonly used continuous and discrete random variables discussed in Chapter 4 can then be introduced. This topic can be covered in 9 lectures. Section 4.5 on extreme value distributions can be skipped if desired. The discussion in Chapter 5 on determining the underlying distribution for a given set of data or observations, and the point and interval estimations of its parameters, can be

covered in 6 lectures. Estimating randomness in response variables including the regression analysis can be covered in 6 lectures. The concept of reliability evaluation, presented in Chapter 7, needs to be covered in detail and may take over 8 lectures. The advanced topics of reliability analysis, discussed in Chapter 8, can be skipped for an introductory undergraduate course. The chapter on simulation, Chapter 9, may take about 4 lectures. These recommendations cover about 43 lectures. Two additional lecture hours are left for examinations or other activities.

Senior or First-Year Graduate Level Course (Average Duration 15 Weeks)

It is expected that these students are familiar with set theory and have some basic knowledge on statistics, and these topics only need to be reviewed. We suggest covering the first 3 chapters in 6 lectures. Chapter 4, with more discussion on Section 4.5, can be covered in 8 lectures. Chapter 5 needs about 6 lectures. About 6 lectures can be spent on Chapter 6. Chapters 7 and 8 need to be covered in detail and may take over 12 lectures. Chapter 9 can be covered in 5 lectures. Again, this lecture guide covers 43 lecture hours; an additional 2 lecture hours are left for examinations and other activities.

Chapter 1

Basic Concept of Reliability

1.1 INTRODUCTORY COMMENTS

The concept of reliability, probability, or risk has been a thought-provoking subject for a long time. Pierre Simon, Marquis de Laplace (1749–1827), published a volume of work on pure and applied mathematics. He also wrote a commentary on general intelligence, published as "A Philosophical Essay on Probabilities." In it, Laplace wrote, "I present here without the aid of analysis the principles and general results of this theory, applying them to the most important questions of life, which are indeed for the most part only problems of probability. Strictly speaking it may even be said that nearly all our knowledge is problematical; and in the small number of things which we are able to know with certainty, even in the mathematical sciences themselves, the principal means of ascertaining truth—induction, and analogy—are based on probabilities; so that the entire system of human knowledge is connected with the theory set forth in this essay." He concluded, "It is seen in this essay that the theory of probabilities is at bottom only common sense reduced to calculus; it makes us appreciate with exactitude that which exact minds feel by a sort of instinct without being able ofttimes to give a reason for it. It leaves no arbitrariness in the choice of opinions and sides to be taken; and by its use can always be determined the most advantageous choice. Thereby it supplements most happily the ignorance and weakness of the human mind" (translation, 1951).

These timeless remarks sum up the importance of probability or reliability concepts in human endeavor. The aim of this book is to present practical methods that apply these concepts to the design of engineering systems.

1.2 WHAT IS RELIABILITY?

Most observable phenomena in the world contain a certain amount of uncertainty; that is, they cannot be predicted with certainty. In general, repeated measurements of physical phenomena generate multiple outcomes. Among these multiple outcomes, some

outcomes are more frequent than others. The occurrence of multiple outcomes without any pattern is described by terms such as *uncertainty*, *randomness*, and *stochasticity*. The word *stochasticity* comes from the Greek word *stochos*, meaning uncertain. For example, if several "identical" specimens of a steel bar were loaded until failure in a laboratory, each specimen would fail at different values of the load. The load capacity of the bar is therefore a random quantity, formally known as a random variable. In general, all the parameters of interest in engineering analysis and design have some degree of uncertainty and thus may be considered to be random variables. Although other methods exist for treating uncertainties, as discussed in Section 1.5.1.2, only probabilistic methods are included in this book.

The planning and design of most engineering systems utilize the basic concept that the capacity, resistance, or supply should at least satisfy the demand. Different terminology is used to describe this concept depending upon the problem under consideration. In structural, geotechnical, and mechanical engineering, the supply can be expressed in terms of resistance, capacity, or strength of a member or a collection of members, and demand can be expressed in terms of applied loads, load combinations, or their effect. For a construction and management project, the completion time is an important parameter in defining success. The estimated completion time during the bidding process and the actual time spent to complete the project will give the essential components of supply and demand. In environmental engineering, the actual air or water quality of a given city or site is always measured with respect to allowable or recommended values suggested by a responsible regulatory agency, for example, the Environmental Protection Agency (EPA), giving the essential components of supply and demand. In transportation engineering, an airport or highway is designed considering future traffic needs; the capacity of the airport or highway must meet the traffic demand. In hydraulics and hydrology engineering, the height and location of a dam to be built on a river may represent the capacity. The annual rainfall, catchment areas and vegetation in them, other rivers or streams that flow into the river being considered, usage upstream and downstream of the river, and location of population centers may represent demand. Errors can never be avoided in surveying projects. The measurement error could be positive or negative, and the quality or sophistication of the equipment being used and the experience of the surveyor may represent capacity. In this case, the acceptable tolerance may indicate demand.

The point is that no matter how supply and demand are modeled or described, most engineering problems must satisfy the concept. However, most of the parameters related to supply and demand are random quantities. In the subsequent chapters, this uncertainty or randomness will be identified and quantified. The primary task of planning and design is to ensure satisfactory performance, that is, to ensure that the capacity or resistance is greater than demand during the system's useful life.

In view of the uncertainties in the problem, satisfactory performance cannot be absolutely ensured. Instead, assurance can only be given in terms of the probability of success in satisfying some performance criterion. In engineering terminology, this probabilistic assurance of performance is referred to as *reliability*.

An alternative way to look at the problem is to consider unsatisfactory performance of the system. In that case, one might measure the probability of failure to satisfy some performance criterion, and the corresponding term would be *risk*. Thus, *risk* and *relia-*

bility are complementary terms. (In some references, the term *risk* is not just the probability of failure but includes the consequence of failure. For example, if the cost of failure is to be included in risk assessment, then risk is defined as the product of the probability of failure and the cost of failure.)

Reliability or risk assessment of engineering systems uses the methods of probability and statistics, between which a distinction needs to be drawn here. Statistics is the mathematical quantification of uncertainty (mean, standard deviation, etc., of a variable; these terms will be defined later), whereas probability theory uses the information from statistics to compute the likelihood of specific events.

1.3 NEED FOR RELIABILITY EVALUATION

Engineers have always recognized the presence of uncertainty in the analysis and design of engineering systems. However, traditional approaches simplify the problem by considering the uncertain parameters to be deterministic and accounting for the uncertainties through the use of empirical safety factors. Safety factors are derived based on past experience but do not absolutely guarantee safety or satisfactory performance. Also, they do not provide any information on how the different parameters of the system influence safety. Therefore, it is difficult to design a system with a uniform distribution of safety levels among the different components using empirical safety factors.

Engineering design is usually a trade-off between maximizing safety levels and minimizing cost. A design methodology that accomplishes both of these goals is highly desirable. Deterministic safety factors do not provide adequate information to achieve optimal use of the available resources to maximize safety. On the other hand, probabilistic analysis does provide the required information for optimum design.

For this reason, several design guidelines or codes have recently been revised to incorporate probabilistic analysis. Examples of such revisions include the American Institute of Steel Construction Load and Resistance Factor Design (1994) specifications and the European and Canadian structural design specifications. Several other design specifications are now in different stages of development to incorporate probabilistic design concepts. The use of probabilistic analysis in these codes is expected to provide more information about system behavior, the influence of different uncertain variables on system performance, and the interaction between different system components. While probabilistic analysis brings rationality to the consideration of uncertainty in design, it does not discount the experience or expertise gathered from a particular system. In fact, the probabilistic methodology includes a "professional factor," which incorporates the expert opinions of experienced designers about different uncertain quantities in the system.

One hurdle in the use of probabilistic methodology is the so-called mathematical sophistication of this approach. However, this perception is really due to a lack of familiarity with probabilistic concepts. Many engineering schools now routinely include these concepts in their undergraduate curriculum. As mentioned in the preface, Accreditation Board of Engineering and Technology (ABET) now requires that all civil engineering undergraduate students in the U.S. take a course in this subject, indicating its importance in civil engineering education. The literature on probabilistic analysis and design and the number of engineering applications have greatly increased in recent years, expanding engineers' familiarity with and acceptance of this methodology.

1.4 MEASURES OF RELIABILITY

Many different terms are used to describe the reliability of an engineering system. Some of the terms are self-explanatory, whereas others are not. The commonly used term *probability of failure* is always associated with a particular performance criterion. An engineering system will usually have several performance criteria, and a probability of failure is associated with each criterion. In addition, an overall system probability of failure may be computed. The probability of failure may be expressed as a fraction, such as 1 in 100, or as a decimal, such as 0.01. Reliability is the probability of successful performance; thus it is the converse of the term *probability of failure*. It is common in the aerospace industry to express reliability in terms of decimals, such as 0.999 or 0.9999, and refer to these numbers as "three 9s reliability" or "four 9s reliability." (The corresponding probability of failure values for these reliability estimates are 0.001 and 0.0001, respectively.)

A measure of reliability, in the context of design specifications, is the safety factor, whose value provides a qualitative measure of safety. The safety factor may be used in the context of the load (or demand) on the system, or the resistance (or capacity) of the system, or both. In the context of the load, the nominally observed value of the load (referred to as the service load) is multiplied by a safety factor greater than 1.0 (referred to as the load factor) to obtain the design load. In the context of the resistance, the nominal value of the resistance of the system is multiplied by a safety factor usually less than 1.0 (referred to as the resistance factor or capacity reduction factor) to obtain the allowable resistance. Both load and resistance are uncertain quantities, with a mean, standard deviation, and so forth. The word *nominal* means that a deterministic value is specified by the designer or manufacturer for the load and/or the resistance for design purposes. In the case of loads, the nominal value is usually above the mean value. In the case of resistance, the nominal value is usually below the mean value. These ideas will be elaborated upon in detail in Chapter 7.

When both load and resistance factors are used, the overall safety is measured by the ratio of values of the load and the resistance. The central safety factor is the ratio of the mean values of the resistance and the load. The nominal safety factor is the ratio of the nominal values of the resistance and the load. Again, these concepts will be elaborated upon in Chapter 7.

For practical structures and performance criteria, it is difficult to compute the probability of failure precisely. Therefore, a first-order estimate frequently is used in probabilistic design specifications. This first-order estimate employs a measure known as the reliability index or safety index (denoted by the Greek symbol beta, β). The concepts of reliability index and first-order approximation of the probability of failure will be described in detail in Chapter 7.

1.5 FACTORS AFFECTING RELIABILITY EVALUATION

Reliability analysis requires information about uncertainties in the system. Before collecting such uncertainty information and proceeding with the reliability analysis, the engineer needs to understand that there are different types of uncertainty in engineering systems, and that each type of uncertainty requires a different approach for data collection and use in reliability evaluation.

1.5.1 Sources of Uncertainty

In a broad sense, uncertainties in a system may come from cognitive (qualitative) and noncognitive (quantitative) sources.

1.5.1.1 Noncognitive Sources

Noncognitive or quantitative sources of uncertainty or randomness can be classified into three types for the purpose of discussion. The first source is the inherent randomness in all physical observation. That is, repeated measurements of the same physical quantity do not yield the same value, due to numerous fluctuations in the environment, test procedure, instruments, observer, and so on. This may be referred to as inherent uncertainty. The engineer tries to address this type of uncertainty by collecting a large number of observations; this provides good information about the variability of the measured quantity and leads to high confidence in the value used in the design. However, the number of observations that can be collected is limited by the availability of resources such as money and time.

This leads to the second source of uncertainty, known as statistical uncertainty. In this case, one does not have precise information about the variability of the physical quantity of interest due to limited data. The information on variability will vary depending on the number of samples used, say 10 or 100. Therefore, quantitative measures of confidence based on the amount of data are added to the reliability evaluation.

A third type of uncertainty is referred to as modeling uncertainty. System analysis models are only approximate representations of system behavior. Computational models strive to capture the essential characteristics of system behavior through idealized mathematical relationships or numerical procedures, for example, finite element methods for structural analysis. In the process, some of the minor determinants of system behavior are ignored, leading to differences between a computational prediction and actual behavior. Probabilistic methodology is able to include modeling uncertainty. Past experience on the difference between computational model and actual behavior can be used to develop a statistical description of modeling error, to be included as an additional variable in the reliability analysis.

These three sources of uncertainty can be illustrated with a simple example. Suppose the wind load or pressure acting on a building needs to be estimated (in units of pounds per square foot). Recorded wind speed data, in miles per hour, can be collected for the site. It is well known that wind speed cannot be predicted with certainty; thus, it is inherently random. Its statistical uncertainty can be estimated by considering past observations, and more data lead to a better estimate. However, the statistical information on wind speed needs to be converted to wind pressure, for which purpose Bernoulli's theorem is commonly used. This introduces another source of uncertainty, known as modeling uncertainty.

1.5.1.2 Cognitive Sources

Cognitive or qualitative sources of uncertainty relate to the vagueness of the problem arising from intellectual abstractions of reality. They may come from (a) the definitions of certain parameters, for example, structural performance (failure or survival), quality,

deterioration, skill and experience of construction workers and engineers, environmental impact of projects, and condition of existing structures; (b) other human factors; and (c) definitions of the interrelationships among the parameters of the problems, especially for complex systems (Ayyub, 1994). These uncertainties are usually dealt with using fuzzy set theory, which is beyond the scope of this book.

1.6 STEPS IN THE MODELING OF UNCERTAINTY

If the sources of uncertainty are noncognitive, and if the uncertainty needs to be quantified using available information, the following essential steps can be taken. Suppose the uncertainty in the annual rainfall or annual maximum wind speed for a particular city needs to be quantified. Obviously, the information can be generated by collecting all the available recorded data on rainfall or wind speed. There could be records of data for the past 50, 75, or 100 years, giving 50, 75, or 100 samples. The necessary statistical information can be extracted from these samples following the steps shown in Figure 1.1. The information collected constitutes the sample space. The randomness characteristics can be described graphically in the form of a histogram or frequency diagram, as will be elaborated upon in Chapter 3. For a more general representation of randomness, the frequency diagram can be approximated by a known theoretical probability density function, such as the normal density function. However, to describe the probability density function uniquely, certain parameters of the distribution need to be estimated. The estimation of these parameters, called statistics, is itself a major com-

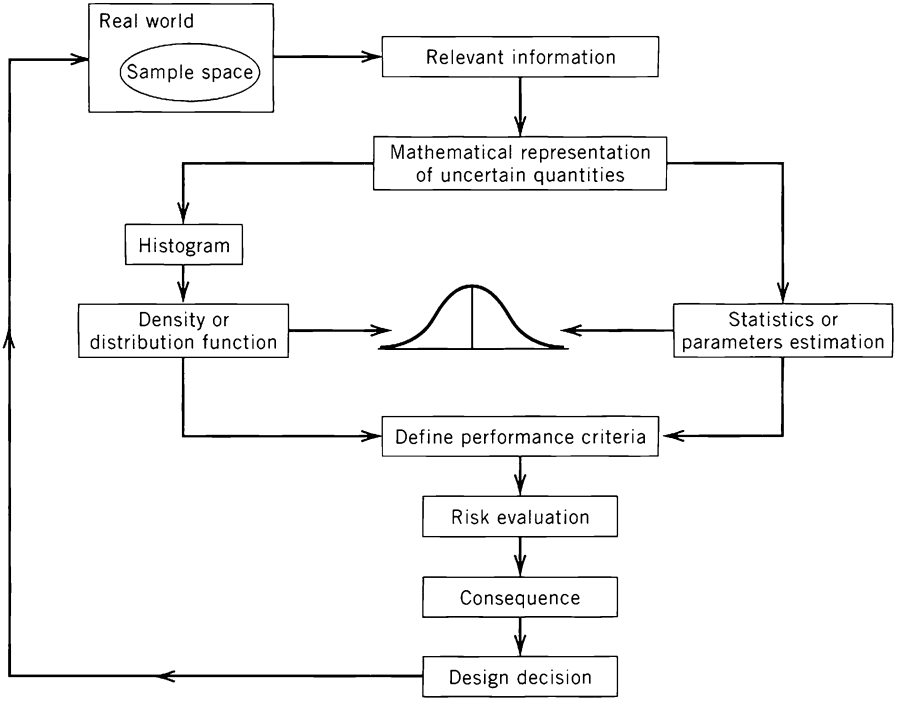


Figure 1.1 Steps in a Probabilistic Study

ponent of the uncertainty analysis. The randomness in each of the load and resistance parameters can be quantified using these statistics. Then, the risk involved in the design can be estimated for a specific performance criterion. All these steps will be discussed in subsequent chapters.

It is undesirable and uneconomical, if not impossible, to design a risk-free structure. In most cases of practical importance, risk can be minimized but cannot be eliminated completely. Nuclear power plants are relatively safer than ordinary buildings and bridges, with a corresponding high cost, but they are not absolutely safe. Making a structure safer costs more money in most cases. For a given structure, the corresponding risks for different design alternatives can be estimated. The information on risk and the corresponding consequences of failure, including the replacement cost of the structure, can be combined using a decision analysis framework to obtain the best alternative. Thus, the probability concept provides a unified framework for quantitative analysis of uncertainty and assessment of risk as well as the formulation of trade-off studies for decision making, planning, and design considering the economic aspects of the problem.

A large number of data are important to accurately implement the risk-based design concept. It is always preferable to estimate uncertainty using an adequate number of reliable observations. However, in many engineering problems, there are very few available data, sometimes only one or two observations. The probability concept can still be used in this case by combining experience, judgment, and observational data. The Bayesian approach can be used for this purpose. In the classical statistical approach, the parameters are assumed to be constant but unknown, and sample statistics are used as the estimators of these parameters. This requires a relatively large amount of data. In the Bayesian approach, the parameters are considered to be random variables themselves, enabling an engineer to systematically combine subjective judgment based on intuition, experience, or indirect information with observed data to obtain a balanced estimate, and to update the estimate as more information becomes available.

In almost all cases, regardless of the amount of available data, the risk-based design concept can be used successfully.

1.7 CONCLUDING REMARKS

The reliability evaluation of engineering systems occurs at two levels: individual performance criteria and overall system performance. These are referred to as component-level and system-level reliability measures, both of which are discussed in this book. Another consideration in reliability evaluation is the effect of time; that is, the system could become less and less reliable with the passage of time due to fatigue, creep, material degradation, and environmental factors such as corrosion. This book does not include methods for time-variant reliability analysis. This area is a fertile topic for research and the methods are still under development, awaiting rigorous validation.

An important approach to uncertainty analysis is the field of random vibration, which considers the random fluctuations of dynamic loads over their duration. The field of random vibration is well established and is already covered by several excellent texts. Therefore, the present text focuses on methods for estimating reliability under time-invariant loads and resistance.

The load effect of simple engineering components, for example, beams and columns, can be easily analyzed by hand computation. The performance criteria for these components involve simple closed-form expressions in terms of the uncertain quantities in the load effect and resistance related parameters. Reliability analysis can easily be performed in such cases. However, most practical engineering systems involve multiple interacting components, and the analysis requires elaborate numerical procedures. In such cases, the performance criteria cannot be related through closed-form relations to the basic uncertain quantities. Additional procedures have to be devised to perform reliability analysis for such systems, which this book briefly addresses.

Chapters 2 through 8 start with the fundamentals of probability and statistics and present the concepts of reliability analysis, with applications to simple components and systems. The simulation concept is introduced in Chapter 9. Simulation techniques can be used for reliability analysis with elementary knowledge of probability and statistics.

Chapter 2

Mathematics of Probability

2.1 INTRODUCTORY COMMENTS

In the previous chapter, randomness in a parameter was defined as the possibility of more than one outcome; in other words, the actual outcome is to some degree subject to chance. The possible outcomes are usually a range of measured or observed values; moreover, within this range certain values may occur more frequently than others. Thus, the fundamental mathematical formulation of probability theory identifies all possible outcomes for a specific problem and defines events in the context of all these possibilities. This formulation cannot be made using conventional algebra; it is necessary to use *set theory*, another branch of mathematics. The essential features of set theory for risk or reliability analysis of engineering systems are discussed in this chapter.

2.2 INTRODUCTION TO SET THEORY

Consider the following example. Suppose it is necessary to be able to go from Point A to Point B at any time on an emergency basis. Also assume that a car must be used, and that this car may not always be in good working condition. To ensure that it is possible to make the trip, common sense suggests that more than one car should be kept available; the question is, how many cars are necessary? Too many cars will cost a lot of money, and they will not be used most of the time. Also, if the condition of each car has to be assessed each time a trip becomes necessary, it may take so much time that the purpose of keeping many cars will be defeated. How can this problem be approached?

2.2.1 Elements of Set Theory

To simplify the discussion, assume that a car can only be in good (G) operating condition or bad (B) operating condition. If there are two cars, the following situations are

possible: both of them are in good condition, both of them are in bad condition, the first one is good and the second one is bad, or the first one is bad and the second one is good. The total number of possible states for these cars is 4 or 2^2. If there are n cars, the total number of possible states will be 2^n. If there are three cars, the number of possible states is $2^3 = 8$. They can be identified as follows:

Car 1	Car 2	Car 3	
G	G	G	All three cars are in good condition.
G	G	B	Two cars are good and one is bad.
G	B	G	
B	G	G	
G	B	B	One car is good and two cars are bad.
B	G	B	
B	B	G	
B	B	B	All three cars are in bad condition.

This example can be used to describe many aspects of set theory. All eight outcomes collectively constitute the *sample space* for the problem, and each of the eight individual outcomes constitutes a *sample point*. Each sample point precludes the occurrence of the others; thus, they are *mutually exclusive*. All events of interest must be defined in the context of this sample space. Suppose X is a *random variable* representing the number of good cars at a given time. A specific value of X is an *event*. The event $X = 3$ means that all three cars are in good condition, and the event $X = 0$ means none of the cars is in good condition. Similarly, if event $X = 2$ or 1, two of the three cars are in good condition, or only one out of three cars is in good condition, respectively. The event that X is either 3 (GGG) or 0 (BBB) contains only one sample point in the sample space; however, the event that X is either 2 (GGB, GBG, BGG) or 1 (GBB, BGB, BBG) contains three sample points. Thus, an event may contain one or more sample points in the sample space; it is a *subset* of a sample space. If we assume that there is a 50% chance that a car is in good or bad condition, each sample point is *equally likely;* that is, the chance of each sample point is 1/8. Thus, in this special case, the chance of X being either 3 or 0 can be calculated by simply counting the number of sample points in the event (i.e., 1/8), and the chance of X being 2 or 1 is 3/8 for each case, indicating the number of sample points for those events. In reality, of course, if the cars are properly maintained the likelihood of each car being in good condition would be much greater than 50%: the chance of all three cars being in good condition is much greater than the chance of all three cars being in bad condition. If the sample points are not equally likely, the counting technique described above will not work. This will be discussed further later.

This simple example introduces the concept of random variables and events. To summarize the discussion so far, each problem must have a sample space. A sample space consists of sample points that are mutually exclusive. An event needs to be defined in terms of sample points, and each event must contain at least one sample point.

An event that does not contain a sample point is called an *impossible event*. It is an empty set or null set, generally denoted as ϕ. If an event contains all the sample points in the sample space, it is called a *certain event* (i.e., the sample space itself) and is

denoted as S. All the sample points in a sample space not belonging to an event E will belong to a *complementary event*, denoted as $\overline{E}$. If E is an event representing three good cars, it will contain only one sample point (GGG). The complementary event $\overline{E}$ will contain all other seven sample points in the sample space.

In the car example just discussed, the number of discrete sample points can be counted; it is a *discrete sample space*. A discrete sample space could be finite or infinite. The car problem is an example of a finite discrete sample space. Suppose that a coin is to be tossed repeatedly until "heads" is thrown for the first time. Each toss will give a discrete sample point. However, it is not known when heads will occur, and it is theoretically possible that heads may never come up, making this an *infinite discrete sample space*.

In a discrete sample space, the sample points can be counted in integers; this is not the case in a *continuous sample space*. Suppose that the design wind velocity for a particular site is of interest. There are upper and lower limits to the possible wind velocity, and the design velocity should be between these limits. However, the wind velocity need not be an integer and could assume an infinite number of values between the limits. This is a case of a continuous sample space, which is always infinite since the sample points cannot be counted.

EXAMPLE 2.1

In Tucson, Arizona, the major part of the water supply may come from pumping from an underground reservoir and/or from a system of canals drawing water from the Colorado River, commonly known as the Central Arizona Project (CAP). The water is stored for future use. On a given summer day, the pumping of water from the underground reservoir can be 2 m, 3 m, or 4 m, in terms of height in the storage tank. The availability of CAP water is equal to 4 m per day of the storage tank. The demand for water in terms of storage tank height can be 8 m, 9 m, or 10 m. Suppose the water tank has a reserve of 20 m water at the beginning of the day.

(a) What are the possible combinations of water supply and demand in a day?

(b) What are the possible water heights in the storage tank at the end of the day?

SOLUTION

Supply (Pumping + CAP)	*Demand*	*Possible water height* (Supply − Demand + 20)
2 + 4 = 6	8	6 − 8 + 20 = 18
6	9	17
6	10	16
3 + 4 = 7	8	7 − 8 + 20 = 19
7	9	18
7	10	17
4 + 4 = 8	8	8 − 8 + 20 = 20
8	9	19
8	10	18

(a) Possible combinations of supply and demand are (6, 8), (6, 9), (6, 10), (7, 8), (7, 9), (7, 10), (8, 8), (8, 9), and (8, 10).

(b) The possible water heights in the tank are 16, 17, 18, 19, and 20 m.

This example represents a finite discrete sample space.

2.2.2 Venn Diagram

For engineers, drawings or sketches are the primary language in which ideas are expressed. It is therefore desirable to draw a sample space and the sample points or events in it. A Venn diagram can be used for this purpose. For the car example, the Venn diagram can be represented by a rectangle consisting of eight other smaller but equal rectangles as shown in Figure 2.1, representing equally likely sample points. If the sample points are not equally likely, the relative size of the rectangles should be different. Figure 2.1 shows the events $X = 0$, 1, 2, or 3, indicating the number of sample points in each event. A Venn diagram provides a simple but comprehensive pictorial description of the problem.

EXAMPLE 2.2

Consider again the wind velocity example. Wind velocity has a magnitude and direction associated with it. The wind velocity records from an airport can be plotted as shown in Figure 2.2, resulting in another form of Venn diagram. A Venn diagram need not always be a rectangle. In general, the form of a Venn diagram should be selected to present information as simply and clearly as possible while retaining all the important aspects of the problem.

2.2.3 Combinations of Events

So far the discussion has focused on problems involving one event at a time. However, in many practical problems, several events must be combined to obtain the necessary information. In the car example, the real objective is to have at least one good car. Therefore, the events that $X = 1$, 2, or 3 are all acceptable. To obtain the likelihood of having at least one good car, these events need to be combined, giving a result of 7/8 (3/8 + 3/8 + 1/8). To solve such problems, it is necessary to know the combination rules in the context of set theory.

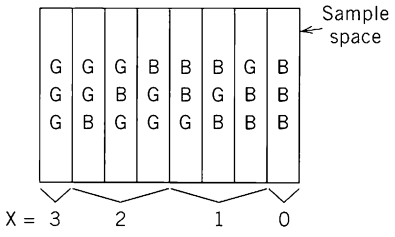


Figure 2.1 Venn Diagram for Car Example

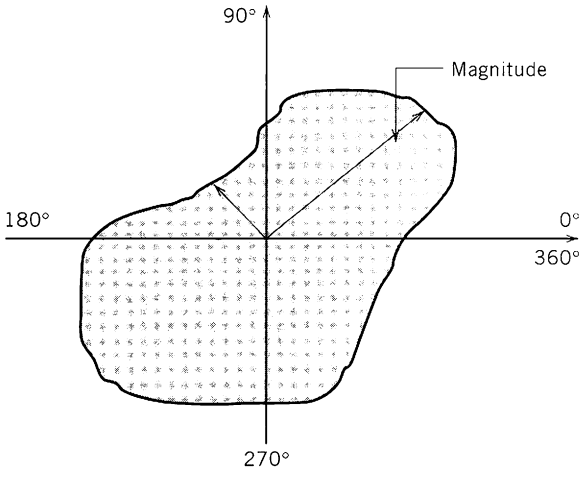


Figure 2.2 Venn Diagram for Wind Velocity

The general concept can be demonstrated with another example. Suppose there is concern about delay in a construction project. Assume E_1 is the event of material shortage, and E_2 is labor shortage. A Venn diagram for these events is shown in Figure 2.3. The shortage of labor or material or both is represented by the shaded area. This is called the *union* of two events, and is denoted by $E_1 \cup E_2$, also referred to as an *OR combination*. Sample points belonging to E_1, E_2, or both will belong to this union of events. The double-shaded, overlapped area in Figure 2.3 indicates shortage of both material and labor (i.e., the joint occurrence of the two events) and is denoted by $E_1 \cap E_2$ or simply $E_1 E_2$, also referred to as an *AND combination*. This is a case of intersection; the sample points common to both E_1 and E_2 constitute the intersection of the two events. In general, the intersection of events will contain fewer sample points than the union of the events.

The area outside the union of the two events in Figure 2.3 indicates that there may be other causes of delay in a construction project, such as bad weather, accidents or worker strikes. This area can be represented as the complement of $E_1 \cup E_2$ and can be denoted as $\overline{E_1 \cup E_2}$. As mentioned earlier, complementary events are points in the sample space not belonging to a specific event.

For many problems, the overlapped area in Figure 2.3 may not exist; that is, there is no common sample point in E_1 and E_2 and the joint occurrence of the events is impossible. When the occurrence of one event precludes the occurrence of the other, they are called *mutually exclusive events*. If E_1 represents survival and E_2 represents

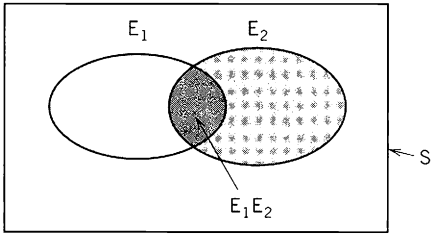
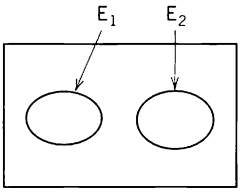
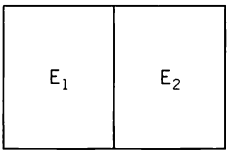


Figure 2.3 Venn Diagram for Delay in a Construction Project



(*a*) Mutually exclusive



(*b*) Collectively exhaustive

Figure 2.4 Mutually Exclusive and Collectively Exhaustive Events

failure of a structure during an earthquake, then the structure will either survive or fail, and the joint occurrence of survival and failure is impossible. The Venn diagram for two mutually exclusive events is shown in Figure 2.4a.

If the union of events, whether mutually exclusive or not, constitutes the entire sample space, as shown in Figure 2.4b, the events are called *collectively exhaustive events;* no other outcomes are possible. If the events are collectively exhaustive, and there is no overlapping between them as shown in Figure 2.1, they are called *mutually exclusive and collectively exhaustive events.* Note that an event and its complement are mutually exclusive and collectively exhaustive.

EXAMPLE 2.3

Three steel specimens are subjected to tensile loads. Let E_i denote the event that the ith specimen yields under these loads. Write expressions for the following events:

(a) Yielding occurs in all three specimens.
(b) Yielding occurs only in two specimens.
(c) Yielding occurs only in Specimen 3.

SOLUTION

(a) $E_1 E_2 E_3$
(b) $(E_1 E_2 \overline{E}_3) \cup (E_1 \overline{E}_2 E_3) \cup (\overline{E}_1 E_2 E_3)$
(c) $\overline{E}_1 \overline{E}_2 E_3$

2.2.4 Operational Rules

The previous discussion described some of the basic operational rules of set theory: union, intersection, and complement of events. These concepts are based on whether the sample points are included in a particular combination of events. The algebraic operations of addition, subtraction, multiplication, and division are not applicable to set theory. Some additional operational rules of set theory are discussed briefly next.

Equality of Sets or Events

Two sets or events are equal if both contain exactly the same sample points. This definition leads to the following observations:

$$E_1 \cup E_1 = E_1$$
$$E_1 \cap E_1 = E_1$$
$$E_1 \cup \phi = E_1$$
$$E_1 \cap \phi = \phi$$
$$E_1 \cup S = S$$
$$E_1 \cap S = E_1$$
$$E_1 \cup \overline{E}_1 = S$$
$$E_1 \cap \overline{E}_1 = \phi$$
$$\overline{\overline{E}}_1 = E_1$$

Set theory has other operational rules, such as the *commutative rule, associative rule* and *distributive rule*, which are similar to the rules for algebraic operations such as addition and subtraction. The commutative rules states that in the union and intersection of two sets, the same results will be obtained regardless of which set is considered first; that is, $E_1 \cup E_2 = E_2 \cup E_1$ or $E_1 E_2 = E_2 E_1$. The associative rule indicates that in the union and intersection of more than two sets, the same results will be obtained regardless of the order in which the sets are combined, that is $(E_1 \cup E_2) \cup E_3 = E_1 \cup (E_2 \cup E_3)$ or $(E_1 E_2) E_3 = E_1 (E_2 E_3)$. The union and intersection of sets are also distributive; that is, $(E_1 \cup E_2) E_3 = E_1 E_3 \cup E_2 E_3$. With the help of Venn diagrams, it can be shown that whether the left- or right-hand side of the equality is considered, the sample points remain the same.

2.2.5 De Morgan's Rule

In many engineering applications, it is necessary to evaluate the union of many sets or events. However, this can be difficult if the number of events is large. De Morgan's rule can be used to simply the problem significantly.

Suppose it is necessary to estimate the risk or reliability of a statically determinate truss consisting of five members as shown in Figure 2.5. Assume that E_i is the event of failure of the ith member; obviously, its complement $\overline{E}_i$ will be the survival of the ith member. The truss will fail if any one of the five members fails, that is, $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$. The complement of truss failure, that is, $\overline{E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5}$ or the survival of the truss, means all five members must survive, or $\overline{E}_1 \overline{E}_2 \overline{E}_3 \overline{E}_4 \overline{E}_5$. This simple observation helps illustrate de Morgan's rule. With set theory, it can be generalized for n events as

$$\overline{E_1 \cup E_2, \ldots, \cup E_n} = \overline{E}_1 \overline{E}_2, \ldots, \overline{E}_n \tag{2.1}$$

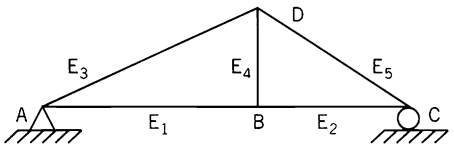


Figure 2.5 Statically Determinate Truss

and

$$\overline{E_1 E_2,\ldots,E_n} = \overline{E}_1 \cup \overline{E}_2,\ldots,\cup\overline{E}_n. \tag{2.2}$$

In simple words, de Morgan's rule states that the complement of the union is equal to the intersection of the respective complements, or the complement of the intersection is equal to the union of the respective complements. This will be discussed in more detail with the help of an example in Section 2.4.

2.3 AXIOMS OF PROBABILITY

So far, the basic concepts of set theory have been discussed. The discussion will now proceed to the mathematics of probability, that is, using set theory to calculate the probability of failure or survival of an engineering system. The mathematics of probability rest on three basic assumptions or axioms.

Axiom 1

The probability of an event, denoted hereafter as $P(E)$, will always be nonnegative, that is, $P(E) \geq 0$.

Axiom 2

The probability of a certain event or the sample space S is 1.0; that is, $P(S) = 1.0$, or $0 \leq P(E) \leq 1.0$.

Axiom 3

For two mutually exclusive events E_1 and E_2, the probability of their union is equal to the summation of their individual probability. Mathematically, this can be expressed as

$$P(E_1 \cup E_2) = P(E_1) + P(E_2). \tag{2.3}$$

As shown in Figure 2.4, this axiom is quite obvious.

Based on these three axioms, several important observations can be made. Since an event E and its complement $\overline{E}$ are mutually exclusive, and $E \cup \overline{E} = S$, then using Axiom 2,

$$P(S) = P(E \cup \overline{E}) = P(E) + P(\overline{E}) = 1.0 \tag{2.4}$$

or

$$P(\overline{E}) = 1.0 - P(E). \tag{2.5}$$

Equation 2.5 indicates that if the probability of failure is known, the reliability or probability of survival can be calculated by subtracting the probability of failure from 1.0; in other words, P(Failure) = 1.0 – P(Survival), or risk = 1.0 – reliability.

Axiom 3 can be generalized for n mutually exclusive events as

$$P(E_1 \cup E_2,\ldots,\cup E_n) = P(E_1) + P(E_2),\ldots,+P(E_n). \tag{2.6}$$

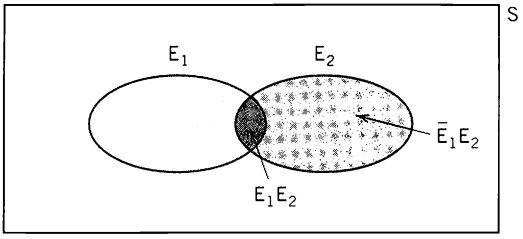


Figure 2.6 Union of Events

Most events are not mutually exclusive; they have some common sample points or overlapping, as Figure 2.6 shows for two events. The third axiom can also be used in this case if all the sample points can be broken into two mutually exclusive sets. In Figure 2.6, although E_1 and E_2 are not mutually exclusive, E_1 and $\bar{E}_1 E_2$ are mutually exclusive and contain the same sample points as $E_1 \cup E_2$. Thus,

$$P(E_1 \cup E_2) = P(E_1 \cup \bar{E}_1 E_2) = P(E_1) + P(\bar{E}_1 E_2). \tag{2.7}$$

Using set theory and Figure 2.6, we can subdivide E_2 into two mutually exclusive events as

$$E_2 = E_1 E_2 \cup \bar{E}_1 E_2. \tag{2.8}$$

Again, using the third axiom,

$$P(E_2) = P(E_1 E_2) + P(\bar{E}_1 E_2). \tag{2.9}$$

If Equation 2.9 is substituted into Equation 2.7, a general expression for calculating the probability of the union of two events can be written as

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 E_2). \tag{2.10}$$

Obviously, if the two events are mutually exclusive, $E_1 E_2 = \phi$, or a null set, $P(E_1 E_2) = 0$, and Equation 2.10 will reduce to Equation 2.3.

The salient features of the set theory discussed in Section 2.2 and the axioms of probability discussed in Section 2.3 can be further clarified with the following examples.

EXAMPLE 2.4 Discrete Sample Space

To increase the efficiency of a design office, managers decide to keep records of the hours required to produce a standard size drawing. It is assumed that the total number of hours required may vary between 60 and 120 hours, in increments of 10 hours. Suppose that a review of 100 drawings provides the following observations:

Total hours	No. of drawings	Frequency
60	4	4/100 = 0.04
70	8	= 0.08
80	12	= 0.12
90	20	= 0.20
100	30	= 0.30
110	17	= 0.17
120	9	= 0.09
	100	1.00

Assume E_1 is the event that a drawing takes between 80 and 100 hours and E_2 the event that a drawing takes more than 90 hours. From the information given in the table, the probabilities of E_1 and E_2 can be calculated as

$$P(E_1) = 0.12 + 0.20 + 0.30 = 0.62$$
$$P(E_2) = 0.30 + 0.17 + 0.09 = 0.56.$$

Similarly,

$$P(E_1E_2) = P(\text{a drawing takes 100 hours}) = 0.30$$

and

$$P(E_1 \cup E_2) = P(\text{a drawing takes at least 80 hours})$$
$$= 0.12 + 0.20 + 0.30 + 0.17 + 0.09 = 0.88.$$

Using Equation 2.10, we can also calculate the probability of the union of the two events as

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1E_2)$$
$$= 0.62 + 0.56 - 0.30 = 0.88.$$

EXAMPLE 2.5 Continuous Sample Space

The preceding example of a discrete sample space can be modified to make it a continuous sample space. A drawing may still take between 60 and 120 hours to produce; however, no direct records are readily available. Suppose that any number of hours between 60 and 120 hours are equally possible. The Venn diagram for the problem is shown in Figure 2.7a. According to Axiom 2, the area under the sample space must be 1.0, as will be elaborated in detail in Section 3.3. Thus, the height of the rectangle is 1/60.

The probability of events E_1 and E_2, as defined earlier for the discrete random variable case, can be shown to be

$$P(E_1) = \frac{1}{60}(100 - 80) = \frac{1}{3} \text{ (see Figure 2.7b)}$$
$$P(E_2) = \frac{1}{60}(120 - 90) = \frac{1}{2} \text{ (see Figure 2.7c).}$$

Similarly,

$$P(E_1E_2) = P \text{ (a drawing takes between 90 and 100 hours)}$$
$$= \frac{1}{60}(100 - 90) = \frac{1}{6} \text{ (see Figure 2.7d).}$$

Thus,

$$P(E_1 \cup E_2) = P \text{ (a drawing takes at least 80 hours)}$$
$$= \frac{1}{60}(120 - 80) = \frac{2}{3} \text{ (see Figure 2.7e).}$$

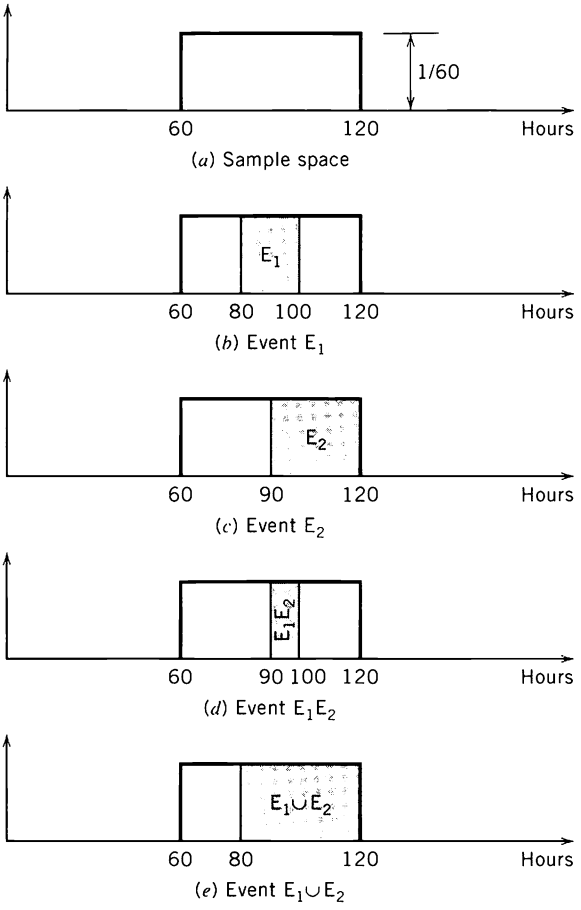


Figure 2.7 Venn Diagram for Hours Required to Produce a Drawing

Using Equation 2.10, we can calculate the probability of the union of the two events as

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1E_2) = \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3}.$$

Both examples demonstrate that the same results can be obtained either by considering all the appropriate sample points or by using the equations of the mathematics of probability (i.e., Equations 2.3 to 2.10).

Extending Equation 2.10 to three events, we can show that

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1E_2) - P(E_2E_3) - P(E_3E_1) + P(E_1E_2E_3). \tag{2.11}$$

Equation 2.11 indicates that as the number of events increases, calculating the probability of their unions becomes increasingly difficult, since the number of terms on the right-hand side of Equation 2.11 can be very large. For practical purposes, de Morgan's rule can be used to help calculate the required probability. Thus, a general expression to calculate the probability of the union of n events can be written as

$$P(E_1 \cup E_2, \ldots, \cup E_n) = 1 - P(\overline{E_1 \cup E_2, \ldots, \cup E_n}) = 1 - P(\overline{E}_1 \overline{E}_2, \ldots, \overline{E}_n). \tag{2.12}$$

If all n events are mutually exclusive, Equation 2.12 becomes

$$P(E_1 \cup E_2, \ldots, \cup E_n) = \sum_{i=1}^{n} P(E_i). \tag{2.13}$$

Equation 2.13 is a more general representation of Equation 2.3 for mutually exclusive events.

2.4 MULTIPLICATION RULE

As discussed in the previous sections, if events are mutually exclusive, the probability of their joint occurrences or intersection of events will be zero. Otherwise, the probability of the intersection of events needs to be calculated using the multiplication rule discussed below. For two events, the multiplication rule can be stated as

$$P(E_1 E_2) = P(E_1 \mid E_2) P(E_2) = P(E_2 \mid E_1) P(E_1). \tag{2.14}$$

$P(E_1 \mid E_2)$ and $P(E_2 \mid E_1)$ are *conditional probabilities*, or the probability of occurrence of one event given that the other has occurred. In a conditional probability calculation, the sample space is changed by conditioning it with respect to the occurrence of one event. The sample space cannot be changed in the middle of the problem; thus, to obtain the probability with respect to the original sample space, the conditional probability must be multiplied by the probability of the event conditioned on, resulting in Equation 2.14.

Sometimes, the occurrence of one event may depend on the occurrence of another event. If E_1 represents runoff and E_2 represents rainfall, then $P(E_1 \mid E_2)$ means the probability of runoff given that there was some rainfall. However, in some cases, the occurrence of one event may not depend on the occurrence of the other. For example, the occurrences of rainfall and earthquake at a site do not depend on each other, and putting a condition on them has no significance. These events are called *statistically independent events*. If E_1 and E_2 are statistically independent events, then $P(E_1 \mid E_2) = P(E_1)$. Thus, Equation 2.14 becomes

$$P(E_1 E_2) = P(E_1) P(E_2). \tag{2.15}$$

The multiplication rule can be summarized as follows. In general, to calculate the probability of the intersection of two events, it is necessary to calculate a conditional probability. However, if the events are statistically independent (the occurrence of one does not depend on the other), the probability of intersection is the product of the individual probabilities. If the events are mutually exclusive (the occurrence of one precludes the occurrence of the other), the probability of their intersection will be zero.

The mathematics of probability for conditioned individual events must also be applicable to their complementary events, and their unions and intersections, as long as the condition is not changed. Thus, it can be easily shown that

$$P(\overline{E}_1 \mid E_2) = 1 - P(E_1 \mid E_2) \tag{2.16}$$

$$P(E_1 \cup E_2 \mid E_3) = P(E_1 \mid E_3) + P(E_2 \mid E_3) - P(E_1E_2 \mid E_3) \tag{2.17}$$

$$P(E_1E_2 \mid E_3) = P(E_1 \mid E_2 \mid E_3)P(E_2 \mid E_3) = P(E_1 \mid E_2E_3)P(E_2 \mid E_3). \tag{2.18}$$

Generalizing the multiplication rule for n events,

$$\begin{aligned} &P(E_1E_2,\ldots,E_n) \\ &= P(E_1 \mid E_2,\ldots,E_n)P(E_2 \mid E_3,\ldots,E_n),\ldots,P(E_{n-1} \mid E_n)P(E_n). \end{aligned} \tag{2.19}$$

If all the events are statistically independent, then

$$P(E_1E_2,\ldots,E_n) = P(E_1)P(E_2),\ldots,P(E_n) = \prod_{i=1}^{n} P(E_i). \tag{2.20}$$

The concept of multiplication rule is illustrated by the following examples.

EXAMPLE 2.6

A building can suffer structural damage by fire or strong earthquakes. Let F and E denote the events, with the corresponding probabilities 0.005 and 0.05, respectively. F and E are statistically independent events. Calculate the probability of structural damage to the building.

SOLUTION

The information in the problem can be summarized as follows: $P(F) = 0.005$, and $P(E) = 0.05$. Using Equation 2.10, we can show that

$$P(\text{structural damage}) = P(F \cup E) = P(F) + P(E) - P(FE).$$

Since F and E are statistically independent events, we can use Equation 2.15 to rewrite the above equation as

$$\begin{aligned} P(F \cup E) &= P(F) + P(E) - P(F)P(E) \\ &= 0.005 + 0.05 - 0.005 \times 0.05 = 0.05475. \end{aligned}$$

EXAMPLE 2.7

A bridge can be damaged by failure in the foundation (F) or in the superstructure (S). The corresponding failure probabilities for a particular bridge are estimated to be 0.05 and 0.01, respectively. Also, if there is foundation failure, then the probability that the superstructure will also suffer some damage is 0.50.

(a) What is the probability of damage to the bridge?

(b) If F and S are statistically independent, what is the probability of damage to the bridge?

SOLUTION

The information in the problem can be summarized as follows: $P(F) = 0.05$, $P(S) = 0.01$, and $P(S \mid F) = 0.5$.

(a) P(damage to the bridge) $= P(F \cup S) = P(F) + P(S) - P(FS) = P(F) + P(S) - P(S \mid F)\,P(F) = 0.05 + 0.01 - 0.5 \times 0.05 = 0.035$.

(b) When F and S are statistically independent events, using Equation 2.15, the probability of damage to the bridge can be calculated as

$$P(F \cup S) = P(F) + P(S) - P(F)P(S)$$
$$= 0.05 + 0.01 - 0.05 \times 0.01 = 0.0595.$$

Comparison of the results in Parts (a) and (b) shows that the probability of damage to the bridge is higher when the two events are statistically independent than when they are dependent.

EXAMPLE 2.8

A community is concerned about its power supply for the coming winter. There are three major sources of power supply, namely electricity, gas, and oil. Let E, G, and O denote the events of shortages of each of these power sources, respectively. Their probabilities are estimated to be 0.15, 0.1, and 0.2, respectively. Furthermore, assume that if there is a shortage in the oil supply, the probability of an electrical power shortage will be doubled, that is, twice the probability of $P(E)$. The shortage of gas may be assumed to be independent of shortages of oil and electricity.

(a) What is the probability that there will be a shortage of all three major sources of power supply?

(b) What is the probability that a shortage will occur in at least one of the following sources: gas, electricity?

(c) What is the probability that exactly two of the three sources of power supply will be short?

(d) If there is a shortage of electricity, what is the probability that gas and oil also will be scarce?

SOLUTION

The information in the problem can be summarized as follows:

$$P(E) = 0.15, \quad P(G) = 0.1, \text{ and } \quad P(O) = 0.2.$$

Also,

$$P(E \mid O) = 2P(E) = 2 \times 0.15 = 0.30.$$

(a) P(shortage of all three major sources of power supply)

$$= P(EGO) = P(EO)P(G) = P(E \mid O)P(O)P(G)$$
$$= 2P(E)P(O)P(G) = 2 \times 0.15 \times 0.2 \times 0.1 = 0.006.$$

(b) P(shortage of at least one of gas and electricity)

$$= P(G \cup E) = P(G) + P(E) - P(GE)$$
$$= P(G) + P(E) - P(G)P(E) = 0.1 + 0.15 - 0.1 \times 0.15 = 0.235$$

(c) P(shortage of exactly two sources out of three)

$$P(EG\overline{O} \cup E\overline{G}O \cup \overline{E}GO) = P(EG\overline{O}) + P(E\overline{G}O) + P(\overline{E}GO)$$
$$P(EG\overline{O}) = P(E\overline{O})P(G) = [1 - P(O \mid E)]P(E)P(G)$$
$$= \left[1 - \frac{P(E \mid O)P(O)}{P(E)}\right]P(E)P(G)$$
$$= \left[1 - \frac{0.3 \times 0.2}{0.15}\right]0.15 \times 0.1 = 0.009$$

$$P(E\overline{G}O) = P(EO)P(\overline{G}) = P(E \mid O)P(O)[1 - P(G)]$$
$$= 0.3 \times 0.2 \times (1 - 0.1) = 0.054$$

$$P(\overline{E}GO) = P(\overline{E}O)P(G) = P(\overline{E} \mid O)P(O)P(G)$$
$$= [1 - P(E \mid O)]P(O)P(G)$$
$$= (1 - 0.3) \times 0.2 \times 0.1 = 0.014$$

Thus,

$$P(EG\overline{O} \cup E\overline{G}O \cup \overline{E}GO) = 0.009 + 0.054 + 0.014 = 0.077.$$

(d) P(shortage of gas and oil given that there is a shortage of electricity)

$$= P(GO \mid E) = \frac{P(EGO)}{P(E)} = \frac{0.006}{0.15} = 0.04$$

This example also shows that if the information about an event is not available in the problem statement it can be derived using the mathematics of probability.

EXAMPLE 2.9 De Morgan's Rule

De Morgan's rule was introduced in Section 2.2.5, and its application is illustrated by the following example. Consider again the five-member truss shown in Figure 2.5. It is assumed that all five members are designed in such a way that the probability of failure of each of them is 10^{-5}, that is, $P(E_i) = 10^{-5}$, $i = 1$ to 5. It is further assumed that the failures of the members are statistically independent of each other. Since the truss is statically determinate, failure of any one member will constitute failure of the entire truss. Therefore, the probability of failure of the entire truss can be calculated as $P(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5)$. As discussed earlier, it is not computationally realistic or convenient to calculate the probability of the union of five events using an equation similar to Equation 2.11 for the case of the union of three events. If the truss contains a large number of members, it would be impractical to calculate the failure probability

in this way. Alternatively, de Morgan's rule can be used for this purpose. For the truss problem under consideration, it can be shown that

$$\begin{aligned} P(\text{failure of the truss}) &= P(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5) \\ &= 1 - P(\overline{E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5}) \\ &= 1 - P(\overline{E}_1\overline{E}_2\overline{E}_3\overline{E}_4\overline{E}_5) \\ &= 1 - P(\overline{E}_1)P(\overline{E}_2)P(\overline{E}_3)P(\overline{E}_4)P(\overline{E}_5) \\ &= 1 - \left(1 - 10^{-5}\right)^5 \approx 5 \times 10^{-5}. \end{aligned}$$

EXAMPLE 2.10

Consider the three events A, B, and C shown in the Venn diagram in Figure 2.8. The following information is available on their probability of occurrence: $P(A) = 0.4$, $P(B) = 0.35$, $P(BC) = 0.15$, and $P(AB) = 0.1$. Determine the probability of the following events:

(a) $A \cup C$
(b) $B \cup C$
(c) $A \cap B \cap C$
(d) $\overline{A} \cup \overline{B}$
(e) $\overline{A \cup B \cup C}$

SOLUTION

The following information can also be extracted from the Venn diagram:

$$P(A \cap C) = P(AC) = 0, \quad P(BC) = P(C) = 0.15.$$

(a) $P(A \cup C) = P(A) + P(C) - P(AC) = 0.4 + 0.15 - 0 = 0.55.$

(b) $P(B \cup C) = P(B) + P(C) - P(BC) = P(B) = 0.35.$

(c) $P(ABC) = P(\phi) = 0.$

(d) Using de Morgan's rule (Equation 2.2), $P(\overline{A} \cup \overline{B}) = P(\overline{AB}) = 1 - P(AB) = 1 - 0.1 = 0.9.$

(e) $$\begin{aligned} P(\overline{A \cup B \cup C}) &= 1 - P(A \cup B \cup C) \\ &= 1 - [P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)] \\ &= 1 - (0.4 + 0.35 + 0.15 - 0.1 - 0.15 - 0 + 0) = 0.35. \end{aligned}$$

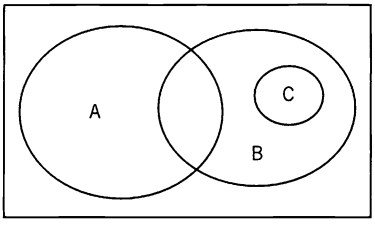


Figure 2.8 Venn Diagram for Example 2.10

2.5 THEOREM OF TOTAL PROBABILITY

Suppose that structural damage (D) to a building can only be caused by three events: fire (F), high wind (W), or earthquake (E). Obviously, D will depend on whether F, W, or E has occurred, and the likelihood of occurrence of F, W, and E. Assume further that F, W, and E are collectively exhaustive and mutually exclusive events. The probability of damage to the building can be calculated as

$$P(D) = P(D \mid F)P(F) + P(D \mid W)P(W) + P(D \mid E)P(E). \tag{2.21}$$

Each term of the right-hand side of Equation 2.21 calculates the probability of damage given that fire, wind, or earthquake has occurred, multiplied by the corresponding probability of occurrence of fire, wind, or earthquake. The concept represented by Equation 2.21 is called the *theorem of total probability.*

The theorem of total probability can be formally presented by the Venn diagram in Figure 2.9. The occurrence of event A depends on the occurrence of other events $E_1, E_1, ..., E_n$, called partitions. They are mutually exclusive (no overlapping), collectively exhaustive (their union constitutes the entire sample space), and $P(E_i) > 0$, $i = 1, 2, ..., n$. Then,

$$A = AS = A(E_1 \cup E_2, \ldots, \cup E_n) = AE_1 \cup AE_2, \ldots, \cup AE_n.$$

It must be noted that the events AE_i are also mutually exclusive. Thus, using Axiom 3,

$$P(A) = P(AE_1) + P(AE_2) + ... + P(AE_n)$$

or

$$P(A) = P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + \ldots + P(A \mid E_n)P(E_n). \tag{2.22}$$

Equation 2.22 is known as the theorem of total probability.

2.6 BAYES' THEOREM

In the context of the theorem of total probability, the probability of occurrence of an event A is calculated, which depends on other mutually exclusive and collectively exhaustive events E_i. Sometimes it is also of interest to know the probability of an event E_i given that A has occurred, an inverse problem. Using the same example considered for the theorem of total probability, and knowing that the building has been damaged, we can use Bayes' theorem to calculate the probability that the damage was

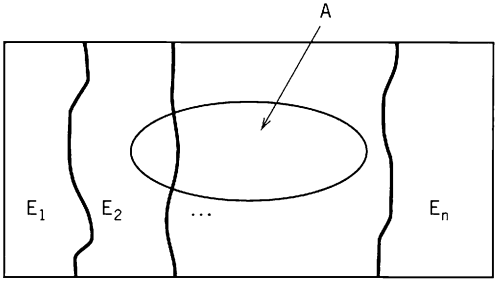


Figure 2.9 Theorem of Total Probability

caused by fire, wind, or earthquake, that is, $P(F \mid D)$, $P(W \mid D)$, or $P(E \mid D)$. This probability can be presented as

$$P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{P(A)} \tag{2.23}$$

or

$$P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{\sum_{i=1}^{n} P(A \mid E_i)P(E_i)}. \tag{2.24}$$

EXAMPLE 2.11

Suppose that air pollution in a city is caused by the following four sources: automobile exhaust (A), industrial exhaust (I), dust (D), and pollen from plants and trees (T). On a particular day, the likelihood of air pollution caused by these sources is 4:3:2:1, respectively. The probability of health hazard (H) caused by these sources is 0.01, 0.005, 0.0005, and 0.0001, respectively. Assume A, I, D, and T are collectively exhaustive and mutually exclusive.

(a) Calculate the probability of health hazard for the city on a given day.

(b) What is the probability that there will be no health hazard in 30 days? In one year (365 days)? Assume health hazard on subsequent days is statistically independent.

SOLUTION

The following information is available: $P(H \mid A) = 0.01$, $P(H \mid I) = 0.005$, $P(H \mid D) = 0.0005$, and $P(H \mid T) = 0.0001$. Also,

$$P(A) = \frac{4}{4+3+2+1} = 0.4.$$

Similarly, $P(I) = 0.3$, $P(D) = 0.2$, and $P(T) = 0.1$.

(a) If the theorem of total probability (Equation 2.22) is used,

$$\begin{aligned} P(H) &= P(H \mid A)P(A) + P(H \mid I)P(I) + P(H \mid D)P(D) + P(H \mid T)P(T) \\ &= 0.01 \times 0.4 + 0.005 \times 0.3 + 0.0005 \times 0.2 + 0.0001 \times 0.1 = 0.00561. \end{aligned}$$

$$\begin{aligned} P(\text{no health hazard in a day}) &= P(\overline{H}) = 1 - 0.00561 \\ &= 0.99439. \end{aligned}$$

(b) Denoting health hazard on the ith day as H_i,

$$\begin{aligned} P(\text{no hazard in 30 days}) &= P(\overline{H}_1\overline{H}_2,\ldots,\overline{H}_{30}) = P(\overline{H}_1)P(\overline{H}_2),\ldots,P(\overline{H}_{30}) = (1-0.00561)^{30} \\ &= 0.84470 \end{aligned}$$

Similarly,

$$P(\text{no hazard in 365 days}) = (1-0.00561)^{365} = 0.12830.$$

EXAMPLE 2.12

A building can have structural damage (D) during its lifetime only from fire (F), high winds (W), and strong earthquakes (E). Thus, F, W, and E are collectively exhaustive events. Further assume that the building will not be structurally damaged simultaneously by F, W, and E, thus making them mutually exclusive events. The probabilities of structural damage to the building if these events occur are estimated to be 0.005, 0.01, and 0.05, respectively. The probabilities of occurrence of F, W, and E during the life of the building are 0.5, 0.3, and 0.2, respectively.

(a) What is the probability that the building will suffer structural damage during its lifetime?

(b) If the building has suffered structural damage, what is the probability that it was caused by F? By W? By E?

SOLUTION

The information in the problem can be summarized as $P(D \mid F) = 0.005$, $P(D \mid W) = 0.01$, and $P(D \mid E) = 0.05$. Also, $P(F) = 0.5$, $P(W) = 0.3$, and $P(E) = 0.2$.

(a) If the theorem of total probability (Equation 2.22) is used, $P(D) = 0.005 \times 0.5 + 0.01 \times 0.3 + 0.05 \times 0.2 = 0.0155$.

(b) If Equation 2.24 is used,

$$P(F \mid D) = \frac{P(D \mid F)P(F)}{P(D)}$$

$$= \frac{0.005 \times 0.5}{0.0155} = 0.16.$$

Similarly,

$$P(W \mid D) = \frac{0.01 \times 0.3}{0.0155} = 0.19$$

and

$$P(E \mid D) = \frac{0.05 \times 0.2}{0.0155} = 0.65.$$

EXAMPLE 2.13

A passenger can travel from home to another city by car (C), ship (S), plane (F), or train (T). In a given year, the passenger made such a trip 80, 15, 100, and 50 times by C, S, F, and T, respectively. The probability of an accident (A) during a trip using these modes of transportation is estimated to be 10^{-5}, 5×10^{-5}, 10^{-6}, and 5×10^{-5}, respectively.

(a) What is the probability of an accident during a trip?

(b) What is the probability of an accident in the next 10 trips?

(c) If there was an accident, what is the probability that the passenger was traveling by car?

SOLUTION

The following information is given:

$$P(C) = \frac{80}{80+15+100+50} = \frac{80}{245} = 0.327.$$

Similarly,

$$P(S) = \frac{15}{245} = 0.061, \quad P(F) = \frac{100}{245} = 0.408, \quad P(T) = \frac{50}{245} = 0.204.$$

Also,

$$P(A \mid C) = 10^{-5}, \quad P(A \mid S) = 5 \times 10^{-5},$$
$$P(A \mid F) = 10^{-6}, \text{ and } P(A \mid T) = 5 \times 10^{-5}.$$

(a) Using the theorem of total probability (Equation 2.22), we can show that

$$\begin{aligned} P(A) &= P(A \mid C)P(C) + P(A \mid S)P(S) + P(A \mid F)P(F) + P(A \mid T)P(T) \\ &= 10^{-5} \times 0.327 + 5 \times 10^{-5} \times 0.061 + 10^{-6} \times 0.408 + 5 \times 10^{-5} \times 0.204 \\ &= 1.6928 \times 10^{-5}. \end{aligned}$$

(b) P(no accident in a trip) = $1 - 1.6928 \times 10^{-5}$.
P(no accident in next 10 trips) = $(1 - 1.6928 \times 10^{-5})^{10}$.
P(accident in next 10 trips) = $1 - (1 - 1.6928 \times 10^{-5})^{10} \approx 1.6928 \times 10^{-4}$.

(c) If Bayes' theorem (Equation 2.23) is used,

$$P(C \mid A) = \frac{P(A \mid C)P(C)}{P(A)} = \frac{10^{-5} \times 0.327}{1.6928 \times 10^{-5}} = 0.193.$$

2.7 REVIEW

The discussions on set theory and mathematics of probability are the foundation for all the subsequent chapters. The examples in this chapter clearly demonstrate that the successful application of set theory and risk assessment requires that events be defined clearly and completely. It is necessary to translate the information available for a particular problem in terms of these events. This may be difficult at first but becomes easier with practice. The risk or reliability can then be calculated using the mathematics of probability.

A short quiz is presented here as a review and to clarify any misunderstandings.

If each of the following statements is always true, circle T; otherwise, circle F. Make sure you understand why each one is true or false.

(1) If events E_1 and E_2 are statistically independent, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$. T F

(2) If events E_1 and E_2 are mutually exclusive, then $P(E_1 \mid E_2) = 0$. T F

(3) If $P(E_1) = P(E_2) = 0.01$, then $P(E_1) \cup P(E_2) = 0.02$. T F

(4) If E_1, E_2, and E_3 are mutually exclusive and collectively exhaustive, then $P(E_1) = 1 - P(E_2) - P(E_3)$. T F

(5) $P(E_1 \mid E_2) = \dfrac{P(E_1)}{P(E_2)}$. T F

(6) $P(\overline{E}_1 \mid \overline{E}_2) = 1 - P(E_1 \mid E_2)$. T F

(7) If E_1 and E_2 are mutually exclusive, then $P(E_1 \mid E_2) = P(E_1)$. T F

(8) $P(\overline{E}_1 \mid E_2 \cup E_3) = 1 - P(E_1 \mid \overline{E_2 \cup E_3})$. T F

(9) $E \cup \overline{E} = S$. T F

(10) $P(E_1 \mid E_2)P(E_2) = P(E_2 \mid E_1)P(E_1)$. T F

2.8 CONCLUDING REMARKS

The concept of set theory, the fundamental mathematical formulation required to calculate the probability of failure or reliability, is introduced in this chapter. Every problem with uncertain quantities/events must have a sample space, which is either discrete or continuous. Each possible outcome represents a sample point within the sample space, and each event must contain at least one sample point. Graphical representations of a sample space and the sample points or events are made using Venn diagrams.

In many practical problems, several events need to be combined to obtain the necessary information, requiring combination rules in terms of union and intersection of events. In this context, mutually exclusive, statistically independent, and collectively exhaustive events have also been defined. An event only contains information on the sample points in it; the mathematics of probability described in three axioms are necessary to utilize this information. These axioms lead to the computation of the probability of unions and intersections of events, the multiplication rule, the theorem of total probability, and Bayes' theorem. It is extremely important that the reader understand these concepts, since they are the basis of all discussions in the following chapters.

2.9 PROBLEMS

2.1 There are two cars in the garage. The operating condition of each car can be described as excellent (E), good (G), or bad (B). Identify all the possible combinations of operating conditions of these two cars (i.e., identify all the sample points). Suppose there are five cars and each car can have the three operating conditions just described. Calculate the total number of sample points. It is not necessary to identify each sample point.

2.2 A 200×200 sq ft land lot is available for a subdivision. Each plot within the subdivision can have a size of 100×100 sq ft (Type A) or 50×50 sq ft (Type B). How many different ways can the lot be subdivided? Do not consider the ordering of lots A and B.

2.3 A site selected for a major construction project has the following wind speed data available. When the wind blows between east (0°) and north (90°), the maximum recorded wind speed is 120 mph; when it blows north (90°) and west (180°), the maximum recorded wind speed is 70 mph; when it blows between west (180°) and south (270°), the maximum wind speed is 110 mph; and when it blows between south (270°) and east (0 or 360°), the maximum wind speed is 80 mph.

(a) Draw a Venn diagram for the sample space.

(b) If an event E is defined as wind speed greater than 100 mph blowing from the north and northeast, show the event on the Venn diagram.

2.4 Suppose the foundation, superstructure, and interior of a residential building have been evaluated by a building inspector. The foundation and the superstructure can be rated as being in good (G) or bad (B) condition. The interior can be rated as excellent (E), good (G), or bad (B). The sample point GGG indicates that the foundation, superstructure, and interior are all in good condition.

(a) Identify all the sample points in the sample space.

(b) For marketing the property, all three items must be in good condition or better. Identify the sample points that make the property marketable.

2.5 A structure is supported on four foundations, namely A, B, C, and D. Suppose Foundation A cannot settle. However, Foundations B, C, and D can have the following amount of settlement:

Foundation B: 0 cm or 1 cm
Foundation C: 0 cm or 2 cm
Foundation D: 0 cm or 3 cm

Sample point (0, 0, 0, 0) indicates that all the four supports did not settle.

(a) Identify all the sample points in the sample space.

(b) Differential settlement between adjacent supports of 2 cm or more will cause excessive cracks in the structure. Identify these sample points.

2.6 The construction of a residential building consists of foundation (F), superstructure (S), plumbing (P), electrical work (E), and painting (N), as shown in Figure P2.6. The possible durations in weeks for each of these activities are shown in the figure. The foundation and the superstructure must be constructed in sequence; however, the plumbing, electrical work, and painting proceed simultaneously.

(a) Identify all the possible completion times of this construction project. (Hint: Calculate completion time for events like FSP, FSE, FSN, and so on).

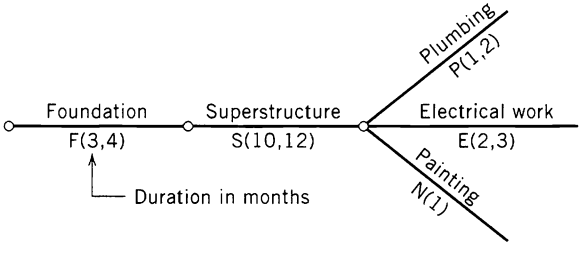


Figure P2.6 Construction Activities

2.7 Two small rivers A and B meet and become a larger river C. For a new housing project, the flood level, defined as the water level above the mean level of river C, is under consideration. Past records indicate that a 1-ft change in the water level of river A causes a 0.5-ft change in the water level of river C, and a 1-ft change in the water level of river B causes a 0.25-ft change in the water level of river C. The flood levels of rivers A and B can be estimated precisely. However, they are equally likely to be 0, 1, 2, and 3 ft for both rivers. Assuming statistical independence of flood levels between rivers A and B, what is the probability that the flood level of river C will exceed 2 ft?

2.8 To study the relationship between the relative density D_r of a sand deposit and the corresponding standard penetration test value or N value, 50 tests were conducted as shown in Figure P2.8. For ease of calculation, the total numbers of test results in each grid are given in the figure. Calculate the following:

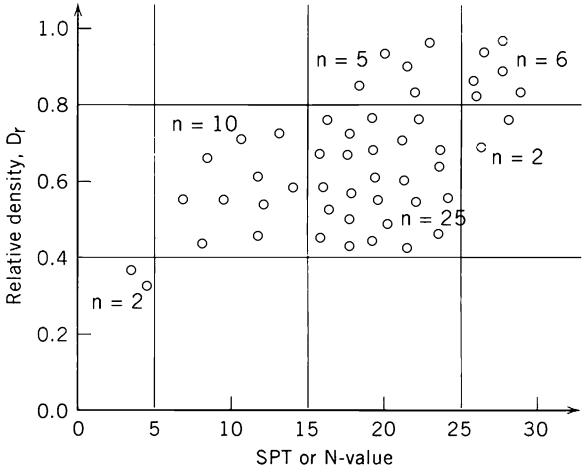


Figure P2.8 Relative Density Versus N-value

(a) $P(0.4 \le D_r)$

(b) $P(D_r > 0.8 \mid N > 15)$

(c) $P(0.4 < D_r \le 0.8 \mid 5 < N \le 25)$

(d) $P(D_r > 0.4 \text{ and } N \le 15)$

(e) $P(D_r \le 0.8 \text{ and } N > 25)$

2.9 At a particular site, the probabilities of occurrence of high wind and moderate earthquake in a single minute are assumed to be 10^{-5} and 10^{-8}, respectively. The occurrence of high wind and moderate earthquake can be assumed to be statistically independent.

(a) According to the building code, the combined effect of these two loads need not be considered. Is this reasonable?

(b) If the events in succeeding minutes are statistically independent, what is the probability that there will be no moderate earthquake in a day? In a year? During the 50-year life of the structure? Hint: Use de Morgan's rule.

2.10 The power supply to a hospital operating room may come from electricity (E) or a diesel generator (D). The diesel generator starts operating only if there is no electricity. The probability that there will be no electricity at any given time is 0.001. If the diesel generator has to supply the power, the probability that it will fail is 0.01.

(a) What is the probability that there will be no power in the operating room?

(b) If there is power in the operating room, what is the probability that both sources of power are in good operating condition?

2.11 A car accident can occur during the winter season due to icy road conditions (I) or lack of visibility (V). The corresponding probabilities of an accident are estimated to be 0.1 and 0.05, respectively. Also, if road conditions are icy, the probability that visibility will also be poor is 0.3.

(a) What is the probability of an accident during the winter?

(b) If I and V are statistically independent, what is the probability of an accident during the winter?

2.12 A reinforced concrete beam can fail due to either excessive bending movement or excessive shear force. The first kind of failure is generally preceded by large deflection (ductile

behavior), whereas the second kind generally occurs suddenly without warning (brittle behavior). Past observations indicate that 5% of the beams that fail are due to shear, whereas 95% are due to bending. Laboratory results indicate that 80% of the beams that fail in shear produce small diagonal cracks just before failure, whereas only 10% of the beams that fail in bending show similar cracks. Suppose that replacement of the beam is justified when shear failure is more likely than bending failure. If some diagonal cracks are observed during an inspection, should the beam be replaced?

2.13 A concentrated load on a cantilever beam may be placed in either location A or B, with probabilities $P(A) = 0.3$ and $P(B) = 0.7$. If the load is placed at A, the probability of bending failure of the beam is 0.01, and the probability of shear failure is 0.001. If the load is placed at B, the probability of bending failure of the beam is 0.02, and the probability of shear failure remains the same. If the beam has shear failure, then the probability of bending failure is 0.9. What is the overall probability of failure of the beam?

2.14 A bolted joint may fail by shearing of the bolt (E_1), bearing between the bolts and plates at the holes (E_2), or tearing at the edges of the plates (E_3). For the joint, the three failure probabilities are estimated to be 0.002, 0.001, and 0.001, respectively. It is unlikely that the bolts will fail in shear if there is a failure due to bearing, tearing, or both. It is certain that the joint will have bearing failure if there is tearing failure. Calculate the probability of failure of the joint.

2.15 Two major cities are connected by a three-lane highway in each direction. Let E_1, E_2, and E_3 denote the right-hand, center, and left-hand lane, respectively. Upon inspection, the maintenance engineer concludes that the probability that each of these three lanes will require major repair work in the next year are 0.10, 0.05, and 0.01, respectively. From past experience, the following information is available:

$$P(E_2 \mid E_1) = 0.8,\ P(E_3 \mid E_2) = 0.9,\ P(E_3 \mid E_1) = 0.5,\ \text{and}\ P(E_3 \mid E_1\, E_2) = 0.9$$

(a) What is the probability that the highway in each direction will need major repairs next year?

(b) If the need for repair in each direction is statistically independent, what is the probability that the highway will need major repair next year?

2.16 Piles are needed to support columns in a high-rise building at a particular site. The probability of failure of a pile due to excessive axial load is estimated to be 10^{-4}. Suppose that four such piles are required under each column, and there are 20 identical columns in the building. Assume that the failure of piles are statistically independent of each other.

(a) What is the probability that none of the piles under a column foundation will fail?

(b) What is the probability that none of the columns in the building will suffer damage due to pile failure?

2.17 The annual probability of a damaging fire and a strong earthquake in a subdivision are estimated to be 0.03 and 0.001, respectively. If there is a strong earthquake, the probability that it will cause fire is 0.3. The occurrences of natural fires and earthquakes are statistically independent.

(a) What is the probability that the sudivision will have a fire due to a strong earthquake in a year?

(b) What is the probability that the subdivision will have a fire in a year?

(c) What is the probability that the subdivision will have no fire in the next 10 years?

2.18 The survival of a building during an earthquake depends on the intensity of the earthquake. For simplicity, earthquake intensity is described as low (L), medium (M), or high (H). The

relative frequency of occurrence of earthquakes of these intensities is 0.5, 0.05, and 0.001 per year. The earthquakes are statistically independent. The likelihoods of failure of the building associated with earthquakes of intensities, L, M, and H, are 0.01, 0.2, and 0.70, respectively.

(a) What is the probability that an earthquake is of low intensity? Moderate intensity? High intensity?

(b) What is the probability of failure of the building during an earthquake?

(c) What is the probability that the building will survive 2 earthquakes? What about 100 earthquakes?

2.19 Good performance (obtaining a grade of A+) in this probability class depends on your attendance (A) and completion of assignments (C). The probabilities that you will receive a grade of A+ are 100%, 70%, 50%, and 0%, if you regularly attend and complete the assignments, if you regularly attend but do not complete the assignments, if you do not regularly attend but complete the assignments regularly, and if you neither attend nor complete assignments, respectively. Further assume that if you attend the class regularly, there is a 90% probability that you will complete the assignments. The probability that you will attend the class regularly is 0.95, and the probability that you will complete the assignments is 0.90.

(a) What is the probability that you will receive an A+ in this class?

(b) If you received an A+, what is the probability that you regularly attended the class and completed the assignments?

2.20 Water supply to a city on a given day comes from one of three reservoirs, with a relative likelihood of 1:1:3 for Reservoirs A, B, and C, respectively. The supplied water may contain excessive bacteria 5%, 10%, and 2% of the time, respectively, if it came from Reservoir A, B, or C.

(a) What is the probability that the water supplied to the city will contain excessive bacteria on a given day?

(b) If the water is found to contain no excessive bacteria on a given day, what is the probability that it came from Reservoir A?

2.21 From past records, it is observed that in a typical year (365 days), air pollution in a city was caused by excessive dust alone on 30 days, by car exhaust alone on 60 days, and on 10 days the pollution was caused by both excessive dust and car exhaust. Assume that they are the only two sources of air pollution and are statistically independent. The probabilities of excessive dust and car exhaust are estimated to be 0.01 and 0.05 per day, respectively.

(a) What is the probability that the city will have an air pollution problem on a given day?

(b) What is the probability that the city will have an air pollution problem on a given day due to dust? Due to car exhaust? Due to both dust and car exhaust?

(c) Suppose that on a particular day the air is polluted. What is the probability that it is caused by car exhaust?

2.22 The probability of a tornado occurring in a subdivision in Tornado Alley in the United States has been estimated to be 0.2 for one occurrence, 0.03 for two occurrences, and 0.001 for three occurrences during the next 50 years. Assume that the chance of four or more occurrences is negligible during the next 50 years. If a tornado hits a structure, the probability that it will suffer damage is 70%. Assume that the damage from multiple tornadoes is statistically independent. What is the probability that a structure will not suffer any damage from tornadoes in the next 50 years?

2.23 A high-rise building is designed against wind load and earthquakes. The designer's calculation shows that during its service life (the next 50 years), the probability that the building

will be damaged only by earthquakes is 0.1, only by windstorms is 0.05, and by both earthquakes and windstorms is 0.2. Past records show that in a given period of 50 years, the probabilities of occurrence of earthquakes and windstorms are 0.2 and 0.5, respectively. Earthquakes and windstorms are statistically independent.

(a) Find the probability that the building will be damaged during its service life (occurrences of earthquakes and windstorms are statistically independent).

(b) If at the end of 50 years damage is found in the building, what is the probability that earthquakes have occurred?

2.24 The probability of occurrence of tornadoes in a county is estimated to be 0.05 per year. Considering the path and width of tornadoes, it is estimated that the chance of a building being hit by a tornado when it strikes the county is 0.01.

(a) What is the probability that a building will be struck by tornadoes in the next 10 years?

(b) Suppose the intensity of tornadoes is classified as strong and weak depending on the maximum wind speed. The likelihoods of strong and weak tornadoes are 1:4. Further assume that if a building is hit by a weak tornado, the probability of damage to the cladding is 0.4; however, the probability of damage to structural members is only 0.04. These probabilities will be doubled if the building is struck by a strong tornado. Also, if the structural members are damaged, the probability of damage to the cladding increases to 0.8, regardless of whether the tornado is strong or weak. Determine the probability of damage to the building when hit by a tornado.

2.25 A city is served by three overnight mail carriers. The market shares for the three carriers, A, B, and C, are 50%, 30%, and 20%, respectively. The past record indicates that they fail to deliver the mail on time 1%, 2%, and 3% of the time, respectively.

(a) What is the probability that an overnight letter will arrive late in that city?

(b) If the overnight letter arrived late, what is the probability that it was sent via A? Via B? Via C?

2.26 Delay (D) in a construction project can be caused by material shortage (M), labor shortage (L), and bad weather (W); the corresponding probabilities are 40%, 40%, and 20%, respectively. Assume M, L, and W are mutually exclusive and collectively exhaustive, and the likelihood of their occurrence is 2:2:1, respectively.

(a) What is the probability of delay of the construction project?

(b) If the project was delayed, what is the probability that the delay was caused by bad weather?

2.27 The safety of a dam depends on the load conditions produced by water levels upstream and downstream. These load conditions can be denoted as dewatering (D), overtopping (O), and flat-top pool condition (F). In a typical year consisting of 365 days, these load conditions can occur on 3, 62, and 300 days, respectively. The probability of failure of the dam in each load condition is 0.001, 0.0001, and 0.000001, respectively. Calculate the probability of failure of the dam in a year. What is the probability of failure in 10 years? In 50 years?

2.28 For mathematical modeling, suppose the major causes of an accident (A) on a segment of highway can be grouped into speeding (S), tiredness (T), carelessness (C), weather conditions (W), and drunkenness (D). The likelihoods of these causes are estimated to be 4:3:1:1:1, respectively. The probabilities of an accident due to S, T, C, W, and D are 0.01, 0.05, 0.01, 0.005, and 0.8, respectively.

(a) Calculate the probability of an accident in the segment of the highway.

(b) If there was an accident, what is the probability that it was caused by drunkenness of the driver?

Chapter 3

Modeling of Uncertainty

3.1 INTRODUCTORY COMMENTS

To develop the mathematics of probability and explain them with the help of examples, we assumed in Chapter 2 that the probability of an event was known. In reality, these probabilities are rarely known and need to be estimated. In many cases, estimating the probability of an event may be the most difficult part of the computation. Before the probability of an event can be estimated, the uncertainty in the problem needs to be quantified, which is the subject of this chapter.

3.2 STEPS IN QUANTIFYING RANDOMNESS

As discussed in Chapters 1 and 2, randomness means that more than one outcome is possible; in other words, the actual outcome is to some degree unpredictable. If a physical quantity is of concern, the possible outcomes are usually a range of measured or observed values; moreover, within this range certain values may occur more frequently than others. Mathematical modeling or representation of a random variable is thus a primary task in any probabilistic formulation, which needs to be conducted systematically. Some of the essential steps in quantifying randomness are discussed in the following sections.

3.2.1 Data Collection

In probabilistic design of engineering systems, the future is predicted using information from the past, including experience and judgment. Thus, it is essential to collect the available relevant information from the past, as shown in Figure 1.1 in Chapter 1. The collected information will constitute the sample space for the random variable under consideration.

The Golden Gate Bridge in San Francisco was built using several types of structural steel members. Forty-one samples were taken, and their yield and ultimate strengths, proportional limit, and Young's modulus were measured (Beard, 1937). Information on Young's modulus is presented in Table 3.1. Similar information can also be obtained for other random variables of interest.

In standard deterministic designs, Young's modulus for steel is usually assumed to be 29,000 ksi. Table 3.1 indicates that this value rarely occurs. Obviously, E should be treated as a random variable, and this randomness must be modeled appropriately.

To make this discussion meaningful, two additional terms, *population* and *sample*, need to be introduced. A population represents all conceivable observations of a random variable. Data collected on Young's modulus for a particular grade of steel from all over the world would represent the population. Since it is impractical to collect the information from all the available sources, a representative sample is collected. Data on Young's modulus given in Table 3.1 represent such a sample. Representative samples are generally used to gather information on population. A relatively large sample size is always preferable.

3.2.2 Descriptors of Randomness

In Table 3.1, the *maximum value* for E is 33,400 ksi and the *minimum value* is 25,900 ksi. The information on minimum and maximum values is very useful, but if one of these values is used for design purposes, it may not be possible to accurately predict

Table 3.1 Young's Modulus E for the Golden Gate Bridge

Test no.	Young's modulus, E (ksi)	Test no.	Young's modulus, E (ksi)
1	28,900	22	25,900
2	29,200	23	32,000
3	27,400	24	33,400
4	28,700	25	30,600
5	28,400	26	32,700
6	29,900	27	31,300
7	30,200	28	30,500
8	29,500	29	31,300
9	29,600	30	29,000
10	28,400	31	29,400
11	28,300	32	28,300
12	29,300	33	30,500
13	29,300	34	31,100
14	28,100	35	29,300
15	30,200	36	27,400
16	30,200	37	29,300
17	30,300	38	29,300
18	31,200	39	31,300
19	28,800	40	27,500
20	27,600	41	29,400
21	29,600		

the actual behavior of the structure, since the actual value of E will probably be something different. The stiffness of the structure will be either underestimated or overestimated. Furthermore, these values may not be the absolute minimum or maximum values. If more samples are collected, the minimum and maximum values may change. Thus, working with minimum or maximum values may not be desirable and usually is not accepted as good design practice.

To overcome the deficiency of the *minimum–maximum* approach, one commonsense approach is to find out the *average* or *mean* or *expected value* of Young's modulus. For the 41 sample points in Table 3.1, the mean value is 29,575.61 ksi. However, the mean value alone does not provide complete information. For example, the mean value of 0 and 100 is 50. The same mean value will be obtained if the numbers are 40 and 60 or 45 and 55. Obviously, information on the *dispersion* of the values with respect to the mean is needed. The measure of dispersion can be expressed in terms of *variance*, *standard deviation*, or *coefficient of variation*. It is also helpful to know if the dispersion is *symmetrical* or *unsymmetrical*. The degree of symmetry can be measured using the concept of *skewness*.

All of these concepts can be expressed mathematically. Suppose X is a random variable and n observations of X are available. The mean or expected value of X, a measure of central tendency in the data, also known as the first central moment and denoted as $E(X)$ or μ_X, can be calculated for the n observations as

$$\text{Mean} = E(X) = \mu_X = \frac{1}{n}\sum_{i=1}^{n} x_i. \tag{3.1}$$

In Equation 3.1, no distinction is made between the population and sample mean; in fact, it is implicitly assumed that the sample size is relatively large. The implication of a small sample size is discussed in Chapter 5.

The variance of X, a measure of spread in the data about the mean, also known as the second central moment and denoted hereafter as $\text{Var}(X)$, can be estimated as

$$\text{Variance} = \text{Var}(X) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_X)^2 \tag{3.2}$$

If the random variable X is expressed in ksi, then obviously the unit of variance will be in $(\text{ksi})^2$. This dimensional problem can be avoided by taking the square root of the variance. This is the standard deviation, denoted as σ_X hereafter, and can be calculated as

$$\sigma_X = \sqrt{\text{Var}(X)} \tag{3.3}$$

Although the standard deviation value is expressed in the same units as the mean value, its absolute value does not clearly indicate the degree of dispersion in the random variable, without referring to the mean value. For example, the value of the standard deviation could be 10 or 100 without indicating the degree of dispersion. Since the mean and the standard deviation values are expressed in the same units, a nondimensional term can be introduced by taking the ratio of the standard deviation and the mean. This is called the coefficient of variation (COV) and will be denoted as $\text{COV}(X)$ or δ_X. Thus,

$$\text{COV}(X) = \delta_X = \frac{\sigma_X}{\mu_X}. \tag{3.4}$$

For a deterministic variable, COV(X) is zero. A smaller value of the COV indicates a smaller amount of uncertainty or randomness in the variable, and a larger amount indicates a larger amount of uncertainty. In many engineering problems, a COV of 0.1 to 0.3 is common for a random variable.

The skewness, also known as the third central moment, can be calculated as

$$\text{Skewness} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_X)^3. \tag{3.5}$$

Since it is a third moment, it can be positive or negative. Again, to avoid dimensional problems, a nondimensional measure of skewness known as the *skewness coefficient* and denoted as θ_X can be introduced as

$$\theta_X = \frac{\text{skewness}}{\sigma_x^3}. \tag{3.6}$$

If θ_X is zero, the randomness is symmetric; if θ_X is positive, the dispersion is more above the mean than below the mean, and if it is negative the dispersion is more below the mean.

With the data given in Table 3.1 and the corresponding equations, the following information can be calculated.

$$\text{Mean} = (1/41)(1,212,600) = 29,575.6 \text{ ksi.}$$

$$\text{Variance} = [1/(41-1)](90,835,609.8) = 2,270,890.2 \text{ (ksi)}^2.$$

$$\text{Standard deviation} = \sqrt{2,270,890.2} = 1,507 \text{ ksi.}$$

$$\text{COV} = 1,507/29,575.6 = 0.051.$$

$$\text{Skewness} = +6.919941834 \times 10^8.$$

$$\text{Skewness coeffecient} = +0.202.$$

Thus, for the given data, the uncertainty in Young's modulus is relatively small, the randomness is unsymmetrical, and the dispersion is more above the mean than below the mean.

3.2.3 Histogram and Frequency Diagram

A preliminary description of the randomness in a variable can be obtained from the numerical values of the parameters just discussed. A more complete description can be obtained by plotting the information graphically in the form of a *histogram*. A histogram for the data in Table 3.1 is shown in Figure 3.1. The following steps are needed to develop the histogram.

Step 1. Arrange the data in increasing order.

Step 2. Subdivide the data into several equal intervals and count the number of observations in each interval.

Step 3. Plot the number of observations in each interval versus the random variable as shown in Figure 3.1, producing a histogram indicating the randomness.

The number or width of the intervals plays an important role in extracting the information on randomness. Suppose only one interval is used; the histogram will look like a big rectangle and will not provide the necessary information on uncertainty. On the

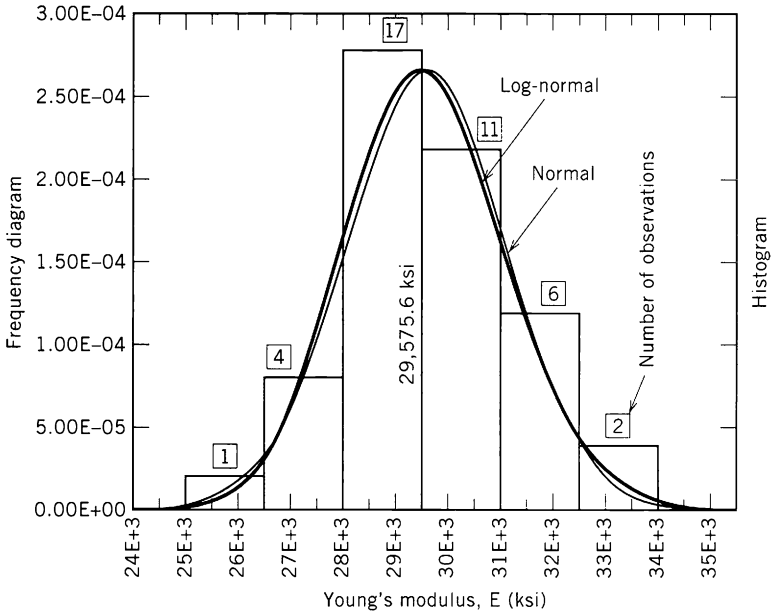


Figure 3.1 Histogram and Frequency Diagram of Young's Modulus

other hand, if a very large number of intervals are used, the width of each interval will be very small and the histogram may look like a series of spikes, defeating the purpose of the histogram. A considerable amount of judgment is necessary to plot a meaningful histogram. An empirical relationship can be used for this purpose:

$$k = 1 + 3.3 \log_{10} n \tag{3.7}$$

where k is the number of intervals and n is the number of samples. For the data shown in Table 3.1, $n = 41$ and so $k = 1 + 3.3 \log_{10} 41 = 6.3$. This gives a rough idea of the number of intervals to use. Suppose a histogram for 1 million data points needs to be plotted. The approximate number of intervals may be calculated as $k = 1 + 3.3 \log_{10} 10^6 = 20.8 \approx 21$. Thus, the number of data points does not necessarily complicate the drawing of a histogram.

Considering the minimum and maximum values of Young's modulus, and rounding them off to 25,000 ksi and 34,000 ksi since they are not the absolute minimum and maximum values, we used six intervals with a width of 1,500 ksi each to develop the histogram shown in Figure 3.1. See also Table 3.2.

The area under a histogram depends on the width of the intervals and the number of data points. For the example under consideration, the area under the histogram will be $1{,}500 \times 41 = 61{,}500$. Since the probability of an event is between 0.0 and 1.0, it will be mathematically advantageous to have the area under a histogram equal to unity. A histogram with a unit area is known as a *frequency diagram.* The frequency diagram can be easily obtained by dividing the ordinates of a histogram by its area. This will not change the shape of the diagram as shown in Figure 3.1. The histogram or frequency diagram will give the relative frequencies of various intervals.

Table 3.2 Data for Histogram and Frequency Diagrams

Interval ($\times 10^3$ ksi)	No. of observations	Fraction of observations
25.0–26.5	1	1/41 = 0.0244
26.5–28.0	4	0.0976
28.0–29.5	17	0.4146
29.5–31.0	11	0.2683
31.0–32.5	6	0.1463
32.5–34.0	2	0.0488
		Σ 1.0000

The area under a frequency diagram can be used to estimate the probability of the event of interest. Suppose that the probability of Young's modulus between 28,000 ksi and 31,000 ksi needs to be calculated. The probability of the event can be estimated as

$$P(28,000 < E \leq 31,000) = \frac{(17+11)1,500}{41 \times 1,500} = 0.6829.$$

One of the primary objectives of a frequency diagram is to fit a curve to the diagram to model the pattern or behavior of the randomness. A curve can be easily fitted to the frequency diagram, as shown in Figure 3.1. As more data are added, the fitted curve will approach the frequency diagram more closely. Attempts can be made to verify whether the fitted curve represents one of many commonly used distributions, such as the normal or lognormal. This will be discussed in detail in Chapter 5.

3.3 ANALYTICAL MODELS TO QUANTIFY RANDOMNESS

The discussion in the previous section needs to be described mathematically. To make this process simple yet comprehensive, discrete and continuous random variables need to be treated separately. Since the Young's modulus example considered in Section 3.1 can be treated as a continuous random variable, it is discussed first.

3.3.1 Continuous Random Variables

From now on, a random variable will be represented in the text by an uppercase letter (e.g, X), and a particular realization of a random variable will be represented by a lowercase letter (e.g., x). The curve shown in Figure 3.1 is called the *probability density function* (PDF) or *density function* and is represented by $f_X(x)$. It does not directly provide information on probability but only indicates the nature of the randomness. To calculate the probability of X having a value between x_1 and x_2, the area under the PDF between these two limits needs to be calculated. This can be expressed as

$$P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)dx. \tag{3.8}$$

To calculate $P(X \leq x)$, which is specifically denoted as $F_X(x)$ and is known as the *cumulative distribution function* (CDF) or simply as *distribution function*, the area under the PDF needs to be integrated for all possible values of X less than or equal to x; in other

words, the integration needs to be carried out theoretically from $-\infty$ to x, and can be expressed as

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(x)dx. \tag{3.9}$$

The CDF directly gives the probability of a random variable having a value less than or equal to a specific value.

Equation 3.8 can be alternatively expressed in terms of the CDF as

$$P(x_1 < X \leq x_2) = \int_{-\infty}^{x_2} f_X(x)dx - \int_{-\infty}^{x_1} f_X(x)dx = F_X(x_2) - F_X(x_1). \tag{3.10}$$

It is obvious that the PDF is the first derivative of the CDF and can be expressed as

$$f_X(x) = \frac{dF_X(x)}{dx}. \tag{3.11}$$

The relationship between the PDF and the CDF is shown conceptually in Figure 3.2 for a continuous random variable. Since the CDF gives information on probability for a random variable, it must satisfy all three axioms of probability discussed in Section 2.3 of Chapter 2. Several important observations can be made from Figure 3.2:

(1) The PDF must be a nonnegative function; it can be zero, and theoretically its range will be from $-\infty$ to $+\infty$.

(2) The CDF must be zero at $-\infty$ and 1.0 at $+\infty$; that is, $F_X(-\infty) = 0.0$, and $F_X(+\infty) = 1.0$.

(3) The CDF is always greater than or equal to zero, that is, $F_X(x) \geq 0.0$, and is a nondecreasing function of a random variable.

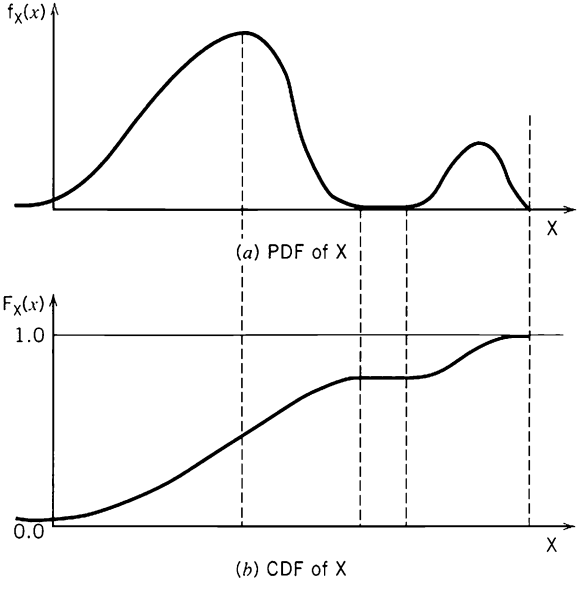


Figure 3.2 PDF and CDF of a Continuous Random Variable

(4) The CDF is continuous with the random variable. In addition, for continuous random variables, the CDF has a derivative.

Any mathematical function that satisfies these requirements can be considered a bona fide CDF and can be used for reliability analysis.

EXAMPLE 3.1

Drawing on experience, an engineer suggests the shape of the PDF of the annual rainfall, R, in a city, as shown in Figure 3.3. The PDF is uniform between 0 and 15 inches of rainfall and is a straight line with a negative slope between 15 and 25 inches. Before anything can be done with this diagram, it must be a bona fide PDF. To satisfy the second axiom of probability, the area under the PDF must be 1.0. Thus, the height h can be estimated from

$$h \times (15 - 0) + 0.5 \times h \times (25 - 15) = 1.0$$

or

$$h = 0.05.$$

Thus, the PDF of the annual rainfall can be expressed mathematically as

$$\begin{aligned} f_R(r) &= 0.05, \quad 0 \le r \le 15 \\ &= 0.125 - 0.005r, \quad 15 \le r \le 25 \\ &= 0.0, \quad \text{elsewhere.} \end{aligned}$$

Once the PDF is defined properly, it is easy to extract other necessary information including the corresponding CDF, as shown in Figure 3.3. It can be shown that

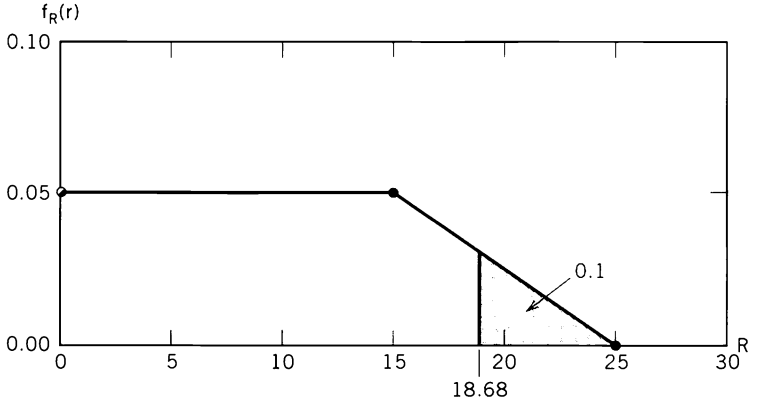
$$\begin{aligned} F_R(r) &= 0, \quad r < 0 \\ F_R(r) &= \int_0^r 0.05dr = 0.05r, \quad 0 \le r \le 15 \\ F_R(r) &= \int_0^{15} 0.05dr + \int_{15}^r (0.125 - 0.005r)dr \\ &= 0.75 + 0.125r - 0.0025r^2 - 0.125 \times 15 + 0.025 \times 15^2 \\ &= -0.5625 + 0.125r - 0.0025r^2, \quad 15 \le r \le 25 \\ F_R(r) &= 1.0, \quad r > 25. \end{aligned}$$

Thus, $P(R \le 20)$ can be calculated in two ways: (a) from the PDF, as

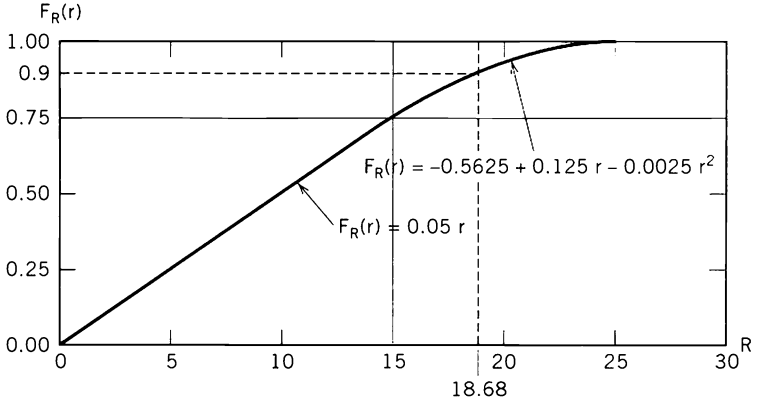
$$\begin{aligned} P(R \le 20) &= \int_0^{15} 0.05dr + \int_{15}^{20} (0.125 - 0.005r)dr \\ &= 0.75 + 0.1875 = 0.9375, \end{aligned}$$

or (b) from the CDF, as

$$F_R(20) = -0.5625 + 0.125 \times 20 - 0.0025 \times 20^2 = 0.9375.$$



(*a*) PDF of annual rainfall, R



(*b*) CDF of annual rainfall, R

Figure 3.3 PDF and CDF of Annual Rainfall

Thus, the probability of an event can be calculated by integrating a probability density function (PDF) or directly using the cumulative distribution function (CDF).

3.3.2 Discrete Random Variables

Considering the physical aspects of some variables used in engineering design, such as the number of fires in a subdivision, the number of strong winds or earthquakes or severe snowstorms, the number of cars crossing an intersection, or the duration in days of a construction activity, it is not logical to model them as continuous random variables. The numbers of such events can only be measured as integers, and they must be treated as discrete random variables. The mathematical treatment of continuous random variables is also applicable to discrete random variables, with some modifications. Since a discrete random variable occurs only at certain discrete points, its relative frequency of occurrence can be evaluated only at these discrete points. This is known as

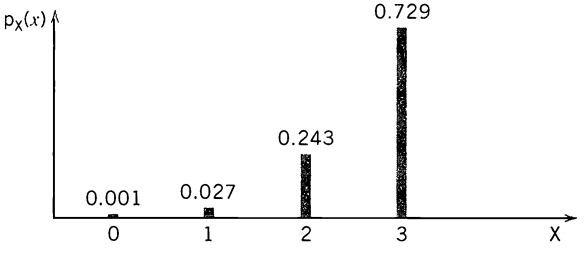
the *probability mass function* (PMF) and is denoted as $p_X(x)$. PMF is similar to PDF; however, it is not a continuous function, and it consists of a series of spikes. The other important difference is that summations of the PMFs are necessary to calculate the CDF for a discrete random variable. The CDF still satisfies all the axioms of probability, but it consists of step functions as shown in Figure 3.4. Mathematically, these observations can be expressed as

$$F_X(x) = P(X \le x) = \sum_{x_i \le x} p_X(x_i). \tag{3.12}$$

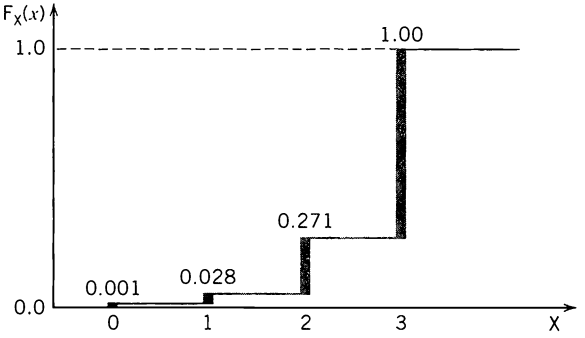
EXAMPLE 3.2

To demonstrate the calculation of PMF and CDF for discrete random variables, we again consider the three-car example from Chapter 2. Assume that a car will be in good condition 90% of the time and in bad condition 10% of the time. Thus, $P(G) = 0.90$ and $P(B) = 0.10$. If we denote X as the random variable representing the number of good cars at a given time, for the problem under consideration, $X = 0$, 1, 2, or 3. The PMFs for these values of X can be calculated as follows:

$$\begin{aligned}
p_X(0) = P(X = 0) &= 0.1 \times 0.1 \times 0.1 &&= 0.001 \\
p_X(1) = P(X = 1) &= 3 \times 0.9 \times 0.1 \times 0.1 &&= 0.027 \\
p_X(2) = P(X = 2) &= 3 \times 0.9 \times 0.9 \times 0.1 &&= 0.243 \\
p_X(3) = P(X = 3) &= 0.9 \times 0.9 \times 0.9 &&= 0.729 \\
&&&\Sigma 1.000
\end{aligned}$$



(*a*) PMF of X



(*b*) CDF of X

Figure 3.4 PMF and CDF of a Discrete Random Variable

The CDF of $X = 2$, that is, $F_X(2)$, can be calculated as

$$P(X \leq 2) = F_X(2) = p_X(0) + p_X(1) + p_X(2) = 0.001 + 0.027 + 0.243 = 0.271.$$

The PMF and the CDF of this example are shown in Figure 3.4.

3.3.3 General Definitions for Uncertainty Descriptors

The procedures for analytically representing the PDF and PMF are now known. They represent the uncertainty in the random variable. To uniquely define these functions, some additional parameters are necessary. Depending upon the nature of the uncertainty, the number of parameters needed could be one or more than one. In most cases, these parameters can be estimated from the information on mean, variance, skewness, etc. of the random variable, which in turn depend on its PDF and PMF. Thus, it is important to develop general expressions for evaluating the mean, variance, skewness, etc. of the random variable from the information on the PDF and PMF.

3.3.3.1 Continuous Random Variables

If X is denoted as a continuous random variable with PDF $f_X(x)$, its mean, also known as the *expected value* of X and denoted as $E(X)$, can be calculated as

$$E(X) = \mu_X = \int_{-\infty}^{\infty} x f_X(x) dx. \tag{3.13}$$

The corresponding variance of X, denoted as $\mathrm{Var}(X)$, is

$$\text{Variance} = \mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx. \tag{3.14}$$

Equation 3.14 can also be shown to be

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x^2 - 2x\mu_X + \mu_X^2) f_X(x) dx$$

or

$$= \int_{-\infty}^{\infty} x^2 f_X(x) dx - 2\mu_X \int_{-\infty}^{\infty} x f_X(x) dx + \mu_X^2 \int_{-\infty}^{\infty} f_X(x) dx$$

or

$$= E(X^2) - 2\mu_X^2 + \mu_X^2 = E(X^2) - \mu_X^2, \tag{3.15}$$

where

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) dx. \tag{3.16}$$

In general, the expected value of any function of X can be calculated as

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx. \tag{3.17}$$

The skewness of X can be calculated as

$$\text{skewness} = \int_{-\infty}^{\infty} (x - \mu_X)^3 f_X(x) dx. \tag{3.18}$$

When these values are known, it is easy to calculate the other parameters, namely, the standard deviation, coefficient of variation, and skewness coefficient, as discussed earlier.

These definitions illustrate the physical meaning of mean, variance, and skewness. Mean is the *centroidal distance* of the area under the PDF from the origin; it is also known as the first moment of the area. Variance is the moment of inertia of the area under the PDF about its mean; it is also known as the second moment of the area under the PDF about its mean. Skewness is the third moment of the area under the PDF about its mean.

EXAMPLE 3.3

Considering the PDF of the rainfall example shown in Figure 3.3 and using Equations 3.13, we can calculate the mean rainfall as follows:

$$E(R) = \int_0^{15} r(0.05) dr + \int_{15}^{25} r(0.125 - 0.005r) dr = 10.208 \text{ in.}$$

Since mean is the centroidal distance of the PDF from the origin, it can also be calculated as

$$\begin{aligned} E(R) &= 0.05 \times 15 \times 7.5 + 0.5 \times 0.05 \times 10 \times (15 + 3.33) \\ &= 5.625 + 4.583 = 10.208 \text{ in.} \end{aligned}$$

When the PDF consists of rectangles and/or triangles, calculating the centroidal distance is simple. However, when the PDF does not consist of simple geometric shapes, calculating the centroidal distance could be complicated. In general, calculation of the mean using Equation 3.13 is preferable.

Using Equation 3.14, the variance of rainfall is

$$\begin{aligned} \text{Var}(R) &= \int_0^{15} (r - 10.208)^2 (0.05) dr + \int_{15}^{25} (r - 10.208)^2 (0.125 - 0.005r) dr \\ &= 19.562 + 17.897 = 37.459 \text{ in.}^2 \end{aligned}$$

As mentioned earlier, the variance can also be estimated by calculating the moment of inertia of the PDF about its mean. For this example, the variance can be calculated as

$$\begin{aligned} \text{Var}(R) &= \frac{1}{3} \times 0.05 \times (10.208^3 + 4.792^3) \\ &\quad + \left[\frac{0.05 \times 10^3}{36} + \frac{1}{2} \times 0.05 \times 10 \times (4.792 + 3.333)^2 \right] = 37.459 \text{ in.}^2 \end{aligned}$$

The result is identical to that obtained using Equation 3.14. In any case,

$$\sigma_R = \sqrt{\text{Var}(R)} = 6.120 \text{ in.}$$

and

$$\text{COV}(R) = \frac{6.120}{10.208} = 0.60.$$

This indicates that the uncertainty in the annual rainfall given by the PDF in Figure 3.3 is large. Similarly,

$$\text{skewness} = \int_0^{15} (r - 10.208)^3 (0.05) dr + \int_{15}^{25} (r - 10.208)^3 (0.125 - 0.005r) dr = +40.676.$$

The corresponding *skewness coefficient* is $40.676/6.120^3 = 0.177$, indicating that the PDF is not symmetrical and there is more spread in the data above the mean than below the mean. For a symmetrical PDF, the skewness coefficient is zero.

3.3.3.2 Discrete Random Variables

If X is a discrete random variable with PMF $p_X(x_i)$, the following expressions can be used to calculate the mean, variance, and skewness.

$$E(X) = \mu_X = \sum_{\text{all } x_i} x_i p_X(x_i) \tag{3.19}$$

$$\text{Variance} = \text{Var}(X) = \sum_{\text{all } x_i} (x_i - \mu_X)^2 p_X(x_i) \tag{3.20}$$

$$\text{skewness} = \sum_{\text{all } x_i} (x_i - \mu_X)^3 p_X(x_i) \tag{3.21}$$

Note that Equations 3.1 and 3.19 are identical. In Equation 3.1, $p_X(x_i)$ is $1/n$, indicating that all the sample points are equally likely.

EXAMPLE 3.4

Consider the three-car problem in Example 3.2. The PMFs for $X = 0$, 1, 2, and 3 are 0.001, 0.027, 0.243, and 0.729, respectively. Using Equations 3.19 to 3.21, the mean, variance, and skewness of X can be estimated as

$$E(X) = 0 \times 0.001 + 1 \times 0.027 + 2 \times 0.243 + 3 \times 0.729 = 2.7$$

$$\text{Var}(X) = (0 - 2.7)^2 \times 0.001 + (1 - 2.7)^2 \times 0.027 + (2 - 2.7)^2 \times 0.243 + (3 - 2.7)^2 \times 0.729 = 0.27$$

$$\sigma_X = 0.52$$

and

$$\text{COV}(X) = \frac{0.52}{2.7} = 0.19$$

$$\text{skewness} = (0 - 2.7)^3 \times 0.001 + (1 - 2.7)^3 \times 0.027 + (2 - 2.7)^3 \times 0.243 + (3 - 2.7)^3 \times 0.729 = -0.216$$

$$\text{skewness coefficient} = -0.216 / 0.52^3 = -1.54.$$

In this case, the PMF is unsymmetrical and there is more spread in the data below the mean than above the mean.

3.3.4 Mode and Median

The discussion on PDF, PMF, and CDF will not be complete without considering two additional parameters, the *mode* and *median* of a random variable X. The mode or modal value of X is the value with the largest PDF or PMF. For a discrete random variable, the mode can be estimated from the PMF diagram by simply observing the value of X with the highest PMF. For a continuous random variable, the mode corresponds to the peak of the PDF, that is, at

$$\frac{df_X(x)}{dx} = 0. \tag{3.22}$$

The modal value of X can be estimated by solving Equation 3.22.

The median of a random variable, x_m, is the value at which the CDF is 0.50, that is, the value of X for which it is equally probable that X will be above or below it. This condition can be expressed as

$$F_X(x_m) = 0.5. \tag{3.23}$$

If we consider the annual rainfall example (Example 3.1) for continuous random variables, the modal value could be any value between 0 and 15 inches, and the median value can be calculated as $0.05r_m = 0.5$, or $r = 10$ inches. In the three-car example (Example 3.2) for discrete random variables, the modal and median values are both 3.

3.3.5 Percentile Value

Selecting the design value of a random variable considering its uncertainty is an important engineering task. It can be done using the concept of *percentile value;* that is, the probability that the actual value will be less than the design value is expressed as a certain percentage. In deterministic designs, the uncertainty in the random variable is considered indirectly by assuming the design value to be a percentile value. In general, for resistance-related random variables, the design value is considered to be less than the 50th percentile; thus, the design value will be less than the median. For load-related random variables, the design value is selected to be greater than the median, so that it is over the 50th percentile value.

EXAMPLE 3.5

Consider the annual rainfall example again. Calculate its 90th percentile value, $r_{0.90}$.

SOLUTION

Mathematically, it can be calculated as

$$P(R \le r_{0.90}) = \int_0^{15} 0.05dr + \int_{15}^{r_{0.90}} (0.125 - 0.005r)dr = 0.90,$$

or

$$0.75 + \left[0.125r - 0.005\frac{r^2}{2}\right]_{15}^{r_{0.90}} = 0.90,$$

or

$$0.0025r_{0.90}^2 - 0.125r_{0.90} + 1.4625 = 0,$$

or

$$r_{0.90} = 18.68 \text{ in.}$$

This is shown in Figure 3.3.

3.4 MULTIPLE RANDOM VARIABLES

So far, the modeling of uncertainty in a single random variable has been discussed. From a practical point of view, it may be necessary to consider more than one random variable to formulate a particular problem. If the load applied to a structure is considered to be a random variable, then the structural response will also be a random variable. The load and the response can be modeled separately as random variables; however, it is more prudent to model the uncertainties jointly, and more information can be extracted from the joint distributions. Thus, it is necessary to extend the discussion to multiple random variables. For the sake of simplicity, the modeling of the joint uncertainty for two random variables will be emphasized in the following sections. The extension of the discussion to more than two random variables will be obvious. Continuous and discrete random variables will be discussed separately.

3.4.1 Joint Distributions

Suppose X and Y are two random variables. If they are continuous, their joint PDF will be denoted as $f_{X,Y}(x, y)$; if they are discrete, their joint PMF will be denoted as $p_{X,Y}(x, y)$. The joint CDF of X and Y for both the continuous and discrete cases can be shown to be

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)dv\ du \tag{3.24}$$

and

$$F_{X,Y}(x,y) = \sum_{x_i \le x}\sum_{y_j \le y} p_{X,Y}(x_i, y_j). \tag{3.25}$$

For the joint distribution, the following observations (similar to Section 3.3) can be made:

(1) The PDF and PMF must be nonnegative.

(2) $F_{X,Y}(-\infty,-\infty) = 0, \quad F_{X,Y}(+\infty,+\infty) = 1.0$

$F_{X,Y}(-\infty, y) = 0, \quad F_{X,Y}(x,-\infty) = 0$

$F_{X,Y}(x,+\infty) = F_X(x), \quad F_{X,Y}(+\infty, y) = F_Y(y)$

(3) The CDF $F_{X,Y}(x, y)$ is always greater than or equal to zero and is a nondecreasing function of the random variables X and Y.

For a single random variable, its PDF or PMF can be plotted on two-dimensional graph paper, as discussed earlier. For two random variables, their PDF or PMF can be described by a three-dimensional plot as shown in Figures 3.5 and 3.6 for continuous and discrete random variables, respectively.

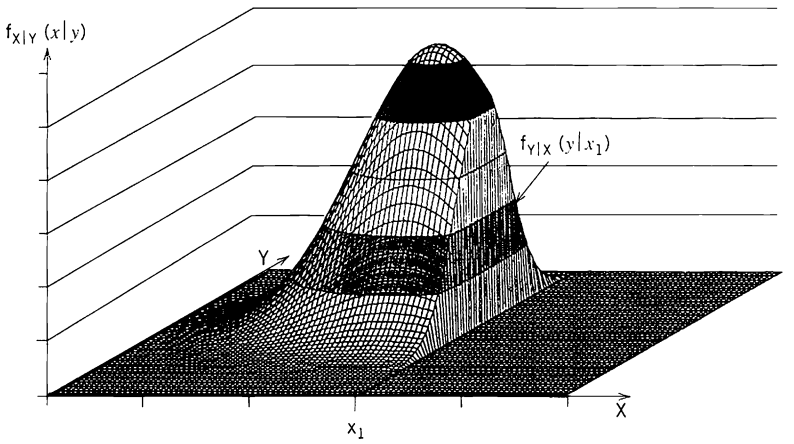


Figure 3.5 Joint PDF of Two Continuous Random Variables

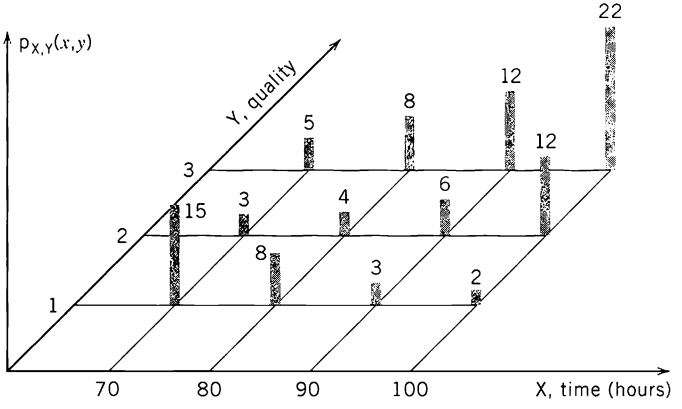


Figure 3.6 Joint PMF of Two Discrete Random Variables

3.4.2 Conditional PDF and PMF

In many engineering problems, the values a random variable takes may be statistically dependent on the values of another random variable. The quality of an engineering drawing may depend on the time spent to develop it. In this situation, it is necessary to calculate the *conditional probability density function* for continuous random variables and the *conditional probability mass function* for discrete random variables. The conditional PDF and PMF can be calculated as

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \tag{3.26}$$

or

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \tag{3.27}$$

and

$$p_{X|Y}(x \mid y) = \frac{P_{X,Y}(x,y)}{P_Y(y)} \tag{3.28}$$

or

$$p_{Y|X}(y \mid x) = \frac{p_{X,Y}(x,y)}{p_X(x)}. \tag{3.29}$$

If X and Y are statistically independent, as discussed in Section 2.4, the condition has no meaning; that is, $f_{X|Y}(x \mid y) = f_X(x)$ or $p_{X|Y}(x \mid y) = p_X(x)$ and it can be shown that

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \tag{3.30}$$

or

$$p_{X,Y}(x,y) = p_X(x) p_Y(y). \tag{3.31}$$

3.4.3 Marginal PDF and PMF

It may be necessary in some cases to calculate the *marginal* PDF or PMF of a random variable, X, from the information on the joint PDF or PMF of X and Y, by completely eliminating the effect of Y. Using the theorem of total probability, we can show the marginal PDF and PMF to be

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \tag{3.32}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \tag{3.33}$$

and

$$p_X(x) = \sum_{\text{all } y_j} p_{X,Y}(x, y_j) \tag{3.34}$$

$$p_Y(y) = \sum_{\text{all } x_i} p_{X,Y}(x_i, y). \tag{3.35}$$

3.4.4 Covariance and Correlation

The problem may become cumbersome if the probability needs to be calculated using the joint distribution of many random variables. Furthermore, the available information may be inadequate to develop the joint distribution of the multiple random variables. For practical applications, it could be advantageous to use the information on the dependence or independence between two random variables to extract as much information as possible. This can be accomplished by covariance and correlation analyses.

Similar to the variance analysis of a single random variable, the *covariance* of two random variables X and Y, denoted as Cov(X,Y), is the second moment about their respective means μ_X and μ_Y, and can be calculated as

$$\begin{aligned} \text{Cov}(X,Y) &= E\left[(X-\mu_X)(Y-\mu_Y)\right] = E\left[XY - \mu_X Y - X\mu_Y + \mu_X\mu_Y\right] \\ &= E(XY) - \mu_X\mu_Y = E(XY) - E(X)E(Y). \end{aligned} \tag{3.36}$$

$E(XY)$ can be calculated as

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dx\,dy. \tag{3.37}$$

If X and Y are statistically independent, then

$$E(XY) = \int_{-\infty}^{\infty} xf_X(x)dx \int_{-\infty}^{\infty} yf_Y(y)dy = E(X)E(Y). \tag{3.38}$$

From Equation 3.36, it can be observed that for statistically independent X and Y, $\text{Cov}(X,Y) = 0$. Otherwise, it can be positive or negative and has the unit that is the square of the unit of the mean. $\text{Cov}(X,Y)$ indicates the degree of *linear relationship* between the two random variables. Nondimensionalizing the covariance will result in the *correlation coefficient*, denoted as $\rho_{X,Y}$, which can be calculated as

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}. \tag{3.39}$$

Values of $\rho_{X,Y}$ range between -1 and $+1$. Again, the correlation coefficient represents the degree of linear dependence between two random variables. The physical characteristics of the correlation coefficient are elaborated in Figure 3.7. Figure 3.7a indicates that there is no linear relationship between the two random variables; the correlation coefficient is expected to be close to zero, and the two random variables can be considered to be uncorrelated. Figure 3.7b indicates a positive relationship between X and Y; that is, Y increases as X increases. However, the relationship is not perfectly linear, indicating that $\rho_{X,Y}$ is expected to be between 0 and 1.0. Figure 3.7e clearly indicates that there could some nonlinear relationship between the two random variables, but since the relationship is not linear, $\rho_{X,Y}$ is expected to be zero.

If the correlation coefficient needs to be calculated from observed sample values, it is rare to obtain values of precisely zero, +1, or −1. The two random variables can be considered to be statistically independent if the correlation coefficient is less than ±0.3; they can be considered to be perfectly correlated if the correlation coefficient is greater than ±0.9.

EXAMPLE 3.6

The water level in a particular lake depends on two sources, direct rainfall X, and inflow from a stream Y. The rainfall Z around the lake can be considered as a random variable with a mean of μ_Z and a standard deviation of σ_Z. X and Y are related to Z as

$$X = a\,Z$$

$$Y = b + c\,Z$$

where a, b, and c are constants. X and Y are functions of a random variable and are therefore also random. Calculate the correlation coefficient $\rho_{X,Y}$.

SOLUTION

The mean and variance of X and Y can be shown to be (see Section 6.2.1)

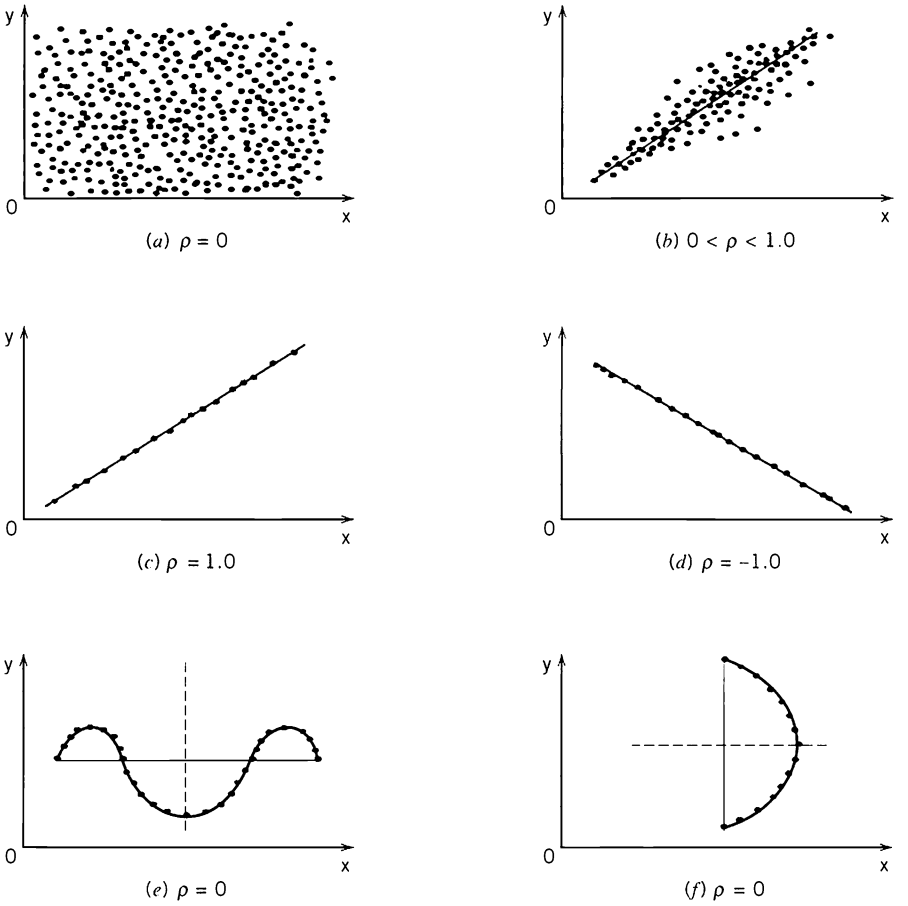


Figure 3.7 Correlation of Two Random Variables

$$\mu_X = a\mu_Z \quad \text{and} \quad \sigma_X^2 = a^2\sigma_Z^2$$
$$\mu_Y = b + c\mu_Z \quad \text{and} \quad \sigma_Y^2 = c^2\sigma_Z^2.$$

Also, $E(XY)$ can be calculated as

$$E(XY) = E[aZ(b+cZ)] = E\left[abZ + acZ^2\right]$$
$$= abE(Z) + acE(Z^2).$$

Using Equation 3.15, for random variable Z, we can show that

$$\sigma_Z^2 = E(Z^2) - \mu_Z^2.$$

Thus,

$$E(XY) = ab\mu_Z + ac\sigma_Z^2 + ac\mu_Z^2.$$

Using Equation 3.36, we can show the covariance of X and Y to be

$$\text{Cov}(X,Y) = ab\,\mu_Z + ac\,\sigma_Z^2 + ac\,\mu_Z^2 - (a\,\mu_Z)(b + c\,\mu_Z) = ac\,\sigma_Z^2.$$

Using Equation 3.39, we can calculate the correlation coefficient of X and Y as

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{ac\sigma_Z^2}{ac\sigma_Z^2} = 1.0.$$

Since both X and Y are linearly related to Z, they are linearly related to each other; therefore, the correlation coefficient of 1.0 between them is expected.

EXAMPLE 3.7

The time to produce a typical engineering drawing, represented by a random variable X, and its quality, represented by a random variable Y, are under consideration. For the sake of discussion, suppose X can be 70, 80, 90, or 100 hours. The quality of a drawing can be considered to be moderate, good, and excellent, and Y can be considered to be 1, 2, and 3, respectively. Suppose that 100 such drawings are evaluated and the information given in Table 3.3 is obtained.

(a) Plot the joint PMF of X and Y.

(b) Plot the marginal PMF of X and Y.

(c) If only excellent quality drawings are acceptable (i.e., $Y = 3$), plot the conditional PMF of X.

(d) Determine the Cov(X,Y) and the corresponding correlation coefficient between X and Y.

Table 3.3 Time and Quality Information on Engineering Drawings

Y \ X	70	80	90	100
1	15	8	3	2
2	3	4	6	12
3	5	8	12	22

SOLUTION

(a) To plot the joint PMF of X and Y, the information can be rearranged as shown below.

X	Y	No. of observations	Relative frequencies
70	1	15	0.15
80	1	8	0.08
90	1	3	0.03
100	1	2	0.02
70	2	3	0.03
80	2	4	0.04
90	2	6	0.06
100	2	12	0.12
70	3	5	0.05
80	3	8	0.08
90	3	12	0.12
100	3	22	0.22

The joint PMF of X and Y is shown in Figure 3.6.

(b) Using Equation 3.34, we can calculate the marginal PMF of X as

$$p_X(70) = 0.15 + 0.03 + 0.05 = 0.23$$
$$p_X(80) = 0.08 + 0.04 + 0.08 = 0.20$$
$$p_X(90) = 0.03 + 0.06 + 0.12 = 0.21$$
$$p_X(100) = 0.02 + 0.12 + 0.22 = 0.36.$$

The marginal PMF of X is plotted in Figure 3.8a. Similarly,

$$p_Y(1) = 0.15 + 0.08 + 0.03 + 0.02 = 0.28$$
$$p_Y(2) = 0.03 + 0.04 + 0.06 + 0.12 = 0.25$$
$$p_Y(3) = 0.05 + 0.08 + 0.12 + 0.22 = 0.47.$$

The marginal PMF of Y is plotted in Figure 3.8b.

(c) Using Equation 3.28, we can show the conditional PMF of X, given $Y = 3$, to be

$$p_{X|Y}(x_i \mid 3) = \frac{p_{X,Y}(x_i, 3)}{p_Y(3)}.$$

Thus,

$$p_{X|Y}(70 \mid 3) = \frac{0.05}{0.47} = 0.11$$
$$p_{X|Y}(80 \mid 3) = \frac{0.08}{0.47} = 0.17$$
$$p_{X|Y}(90 \mid 3) = \frac{0.12}{0.47} = 0.25$$
$$p_{X|Y}(100 \mid 3) = \frac{0.22}{0.47} = 0.47.$$

The conditional PMF of X is shown in Figure 3.8c.

(d) $\mathrm{Cov}(X,Y)$ and the correlation coefficient $\rho_{X,Y}$ can be estimated by using Equations 3.36 and 3.39, respectively. The required information can be calculated as follows:

$$E(X) = 70 \times 0.23 + 80 \times 0.20 + 90 \times 0.21 + 100 \times 0.36 = 87$$
$$\mathrm{Var}(X) = (70-87)^2 \times 0.23 + (80-87)^2 \times 0.20 + (90-87)^2 \times 0.21 + (100-87)^2 \times 0.36 = 139$$
$$\sigma_X = 11.79.$$

Similarly,

$$E(Y) = 1 \times 0.28 + 2 \times 0.25 + 3 \times 0.47 = 2.19$$
$$\mathrm{Var}(Y) = (1-2.19)^2 \times 0.28 + (2-2.19)^2 \times 0.25 + (3-2.19)^2 \times 0.47 = 0.7139$$
$$\sigma_Y = 0.845$$

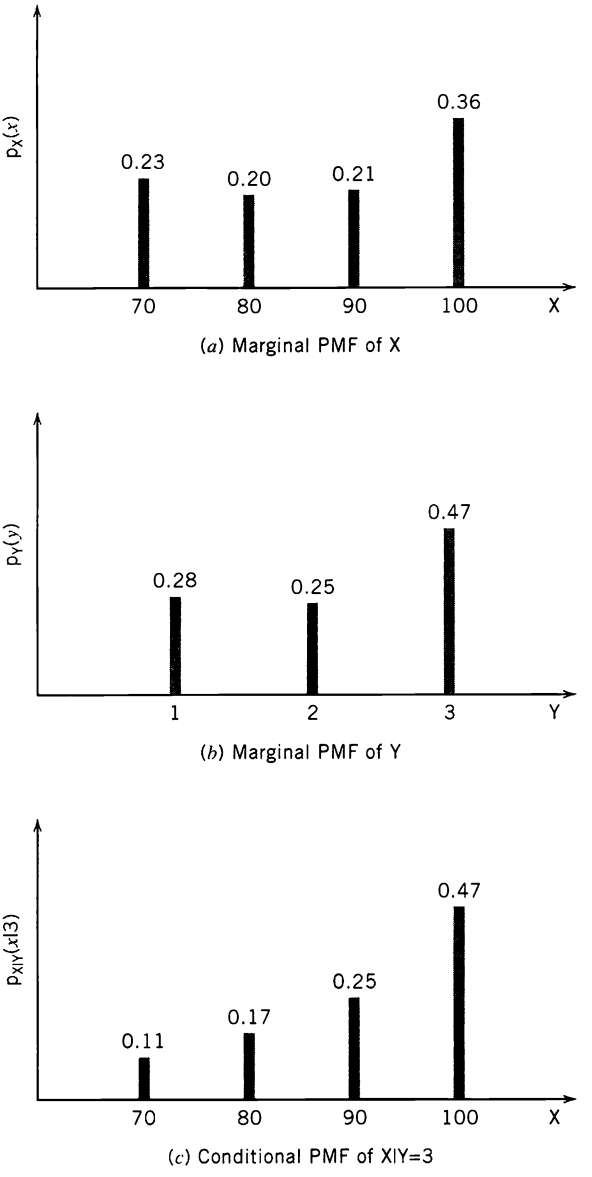


Figure 3.8 Marginal PMF of Time and Quality

$$E(XY) = 70\times1\times0.15 + 80\times1\times0.08 + 90\times1\times0.03 + 100\times 1\times0.02 + 70\times2\times0.03 + 80\times2\times0.04 + 90\times2\times0.06 + 100\times2\times0.12 + 70\times3\times0.05 + 80\times3\times0.08 + 90\times 3\times0.12 + 100\times3\times0.22 = 195.1$$

$$\text{Cov}(X,Y) = 195.1 - 87\times2.19 = 4.57$$

$$\rho_{X,Y} = \frac{4.57}{11.79\times0.845} = +0.46.$$

EXAMPLE 3.8

The joint density function of two random variables X and Y can be represented as

$$f_{X,Y}(x,y) = c(x^2-4)(y^2-9), \quad 0 \le x \le 2 \text{ and } 0 \le y \le 3$$
$$= 0, \quad \text{elsewhere.}$$

(a) Determine the constant c.
(b) Determine the marginal density function for X.
(c) Determine the marginal density function for Y.
(d) Are X and Y statistically independent?
(e) Determine the probability of the following events:
  (1) $P(X > 1 \mid Y = 2)$
  (2) $F_{X,Y}(1, 3)$

SOLUTION

(a)
$$\int_0^3 \int_0^2 c(x^2-4)(y^2-9)dxdy = 1.0$$

or

$$\int_0^3 c(y^2-9)\left[\frac{x^3}{3} - 4x\right]_0^2 dy = \int_0^3 -\frac{16}{3}c(y^2-9)dy = 1.0$$

or

$$-\frac{16}{3}c\left[\frac{y^3}{3} - 9y\right]_0^3 = 1.0$$

or

$$c = \frac{1}{96}$$

(b)
$$f_X(x) = \int_0^3 \frac{1}{96}(x^2-4)(y^2-9)dy = -\frac{3}{16}(x^2-4)$$

(c)
$$f_Y(y) = \int_0^2 \frac{1}{96}(x^2-4)(y^2-9)dx = -\frac{1}{18}(y^2-9)$$

(d)
$$f_X(x)f_Y(y) = \left[-\frac{3}{16}(x^2-4)\right]\left[-\frac{1}{18}(y^2-9)\right]$$
$$= \frac{1}{96}(x^2-4)(y^2-9) = f_{X,Y}(x,y).$$

Thus, X and Y are statistically independent random variables.

(e1) $$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x) = -\frac{3}{16}(x^2 - 4).$$

Thus, $$P(X > 1 \mid Y = 2) = \int_1^2 -\frac{3}{16}(x^2 - 4)dx = -\frac{3}{16}\left[\frac{x^3}{3} - 4x\right]_1^2 = 0.3125.$$

(e2) $$F_{X,Y}(1,3) = \frac{1}{96}\int_0^1 (x^2 - 4)dx \int_0^3 (y^2 - 9)dy = 0.6875.$$

3.4.5 Multivariate Distributions

In general, the explicit consideration of multivariate distributions is mathematically cumbersome. However, standard procedures as outlined in Section 3.4.1 can still be used if the joint PDF of a multivariate distribution is known. Suppose X and Y are jointly normally distributed. The normal distribution will be discussed in detail in Chapter 4. To define the PDF of this bivariate normal distribution requires five parameters, namely, the mean values of X and Y, μ_X and μ_Y, their standard deviations σ_X and σ_Y, and the correlation coefficient $\rho_{X,Y}$. The PDF of the bivariate normal distribution can be expressed as

$$f_{X,Y}(x,y) = \frac{1}{2\pi\,\sigma_X\,\sigma_Y\sqrt{1-\rho_{X,Y}^2}} \exp\left\{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho_{X,Y}\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$$

$$-\infty < x < \infty, \quad -\infty < y < \infty. \tag{3.40}$$

The PDF of a multivariate normal distribution is more complicated.

3.5 CONCLUDING REMARKS

Modeling and quantifying uncertainties in random variables are the initial and essential steps in any risk-based analysis and design. Collecting data and extracting information from the data in terms of many descriptors are introduced in this chapter. Continuous and discrete random variables are considered.

Modeling of multiple random variables and their correlation or dependence on each other are presented. The information presented here is expected to provide sufficient background in modeling and quantifying uncertainties in random variables.

3.6 PROBLEMS

3.1 In an examination for a class of 30 students, the following scores were obtained: 99, 45, 60, 80, 95, 100, 95, 91, 85, 87, 77, 75, 61, 71, 85, 88, 83, 85, 79, 81, 82, 55, 63, 75, 82, 88, 77, 78, 41, and 70.

(a) Draw the histogram for the data.

(b) Draw the frequency diagram for the data.

(c) Calculate the mean, variance, standard deviation, coefficient of variation, skewness, and skewness coefficient for the test scores.

(d) Assume that a student must score at least 85 to receive an A grade. What is the probability that any student in the class will receive an A, using the actual scores only? What will be the corresponding probability if the frequency diagram is used instead?

3.2 The annual precipitation in inches per year during the past 30 years in Tucson, Arizona, is as follows: 11.60, 7.19, 12.69, 11.86, 14.81, 8.07, 11.15, 8.00, 9.55, 11.02, 19.54, 8.63, 12.33, 8.53, 16.55, 19.74, 18.40, 11.37, 10.55, 8.68, 9.62, 6.93, 14.80, 10.64, 14.76, 15.19, 14.56, 9.68, 11.13, and 4.35.

(a) Draw the histogram for the data.

(b) Draw the frequency diagram for the data.

(c) Calculate the mean, variance, standard deviation, coefficient of variation, skewness, and skewness coefficient for the precipitation.

(d) Using the frequency diagram, calculate the probability that the annual precipitation in Tucson will exceed 12 in./yr using the actual data and using the frequency diagram.

3.3 The traveling time from the office to the nearest airport may be 0.5, 1.0, 1.5, 2.0, 2.5, or 3.0 hours depending upon the time of travel. The corresponding PMFs are shown in Figure P3.3. Calculate the following information on the travel time:

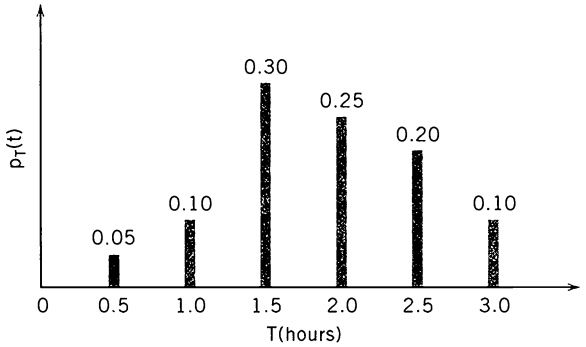


Figure P3.3 PMFs of Travel Time

(a) The mean.

(b) The variance, standard deviation, and coefficient of variation.

(c) The skewness and skewness coefficient.

3.4 In order to bid for a nonstandard construction job, an engineer needs to estimate the duration, D, of the project. Since no prior information on similar jobs is available, the engineer estimates that it may take 10 to 20 days. Suppose the PDF of D can be defined by a uniform distribution between 10 and 20 days.

(a) Define the PDF of D.

(b) Define the CDF of D.

(c) Calculate the mean of D.

(d) Calculate the variance, standard deviation, and coefficient of variation of D.

(e) Calculate the skewness and skewness coefficient of D.

(f) Calculate the modal and median values of D.

3.5 The error X in a measurement is modeled with a probability density function in the shape of a cosine curve:

$$f_X(x) = C\cos\frac{\pi x}{2E_0}, \quad -E_0 \le x \le E_0$$

$$= 0, \quad \text{elsewhere.}$$

(a) Determine the normalizing factor C.

(b) Plot the PDF.

(c) Calculate the CDF and plot it.
(d) What is the probability that X is greater than $E_0/2$?
(e) What is the probability that the error is greater than $E_0/2$ in absolute magnitude?
(f) Calculate the PDF from the CDF obtained in Part c.

3.6 The probability density function of error in measurement associated with surveying for a building project can be described by a second-order parabola as shown in Figure P3.6.

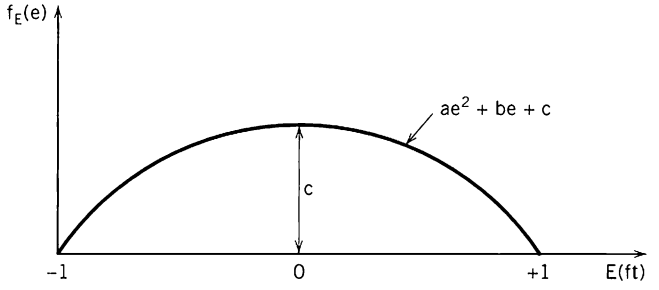


Figure P3.6 PDF of Error

(a) Determine c to make it a bona fide PDF.
(b) Write an expression for its CDF, that is, $F_E(e)$.
(c) What is the probability that the error will be more than 6 inches?

3.7 The PDF of the annual rainfall, R, of a city is shown in Figure P3.7.

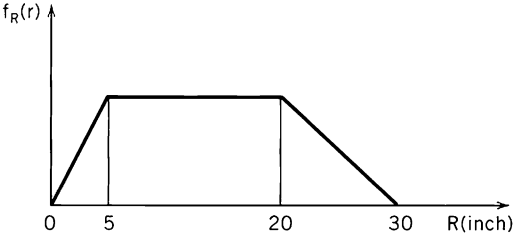


Figure P3.7 PDF of Annual Rainfall

(a) Define the PDF of R properly. Then determine the following:
(b) The mean value of R.
(c) The median of R.
(d) The mode of R.
(e) The variance, standard deviation, and coefficient of variation of R.
(f) The skewness and skewness coefficient of R.

3.8 The CDF of the hourly traffic volume, X, at an intersection can be expressed as

$$F_X(x) = \frac{x^2}{200,000}, \quad 0 \le x \le 400$$

$$= \frac{x}{50} - \frac{x^2}{50,000} - 4, \quad 400 < x \le 500$$

$$= 0, \quad x < 0$$

$$= 1.0, \quad x > 500.$$

(a) Determine the PDF of X and draw it.
(b) Determine the mean, mode, and median of X.

(c) Determine the variance, standard deviation, and coefficient of variation of X.

(d) Determine the skewness and skewness coefficient of X.

3.9 The study duration and grade point average (GPA) of students graduating with a B.S. degree from an engineering school were studied. With X defined as the number of years it takes to graduate and Y as the GPA, it was observed that X could be 4, 5, or 6 years and Y could be 2, 3, or 4. The following table shows the number of students for each combination of X and Y.

Y \ X	4	5	6
2	5	15	60
3	50	80	20
4	20	40	10

(a) Plot the joint PMF of X and Y.

(b) Plot the marginal PMF of X and Y.

(c) If only a GPA of 3 is under consideration (i.e., $Y = 3$) plot the conditional PMF of X.

(d) Determine $\text{Cov}(X, Y)$ and the corresponding correlation coefficient between X and Y.

3.10 A person's commuting time from home to the workplace (X) and from the workplace to home (Y) is studied for 100 days. Assume that the commuting time each way can be approximated as 30, 40, or 50 minutes. The following table shows the number of days for each combination of X and Y.

Y \ X	30	40	50
30	10	20	25
40	5	30	4
50	3	2	1

(a) Plot the joint PMF of X and Y.

(b) Plot the marginal PMF of X and Y.

(c) Considering 30 minutes commuting time from home to workplace (i.e., $X = 30$), calculate the conditional PMF of Y.

(d) What is the probability that the commuting time in each direction on a particular day will be at least 40 minutes?

(e) Determine $\text{Cov}(X, Y)$ and the corresponding correlation coefficient.

3.11 The joint probability density function of two random variables X and Y can be represented as

$$f_{X,Y}(x,y) = c(x^2 + xy + y^2), \quad 0 \le x \le 2 \text{ and } 2 \le y \le 4$$
$$= 0, \quad \text{elsewhere.}$$

(a) Determine the constant c.

(b) Determine the marginal density function for X.

(c) Determine the marginal density function for Y.

(d) Are X and Y statistically independent?

(e) Determine the probability of the following events:

(i) $P(Y > 3 \mid X = 1)$

(ii) $F_{X,Y}(1, 3)$

3.12 The joint probability density function of two random variables X and Y can be represented as

$$f_{X,Y}(x, y) = ce^{x+y}, \quad 0 \le x \le 1 \text{ and } 0 \le y \le 2$$
$$= 0, \quad \text{elsewhere.}$$

(a) Determine the constant c.

(b) Determine the marginal density function for X.

(c) Determine the marginal density function for Y.

(d) Are X and Y statistically independent?

(e) Determine the probability of $P(X \le 0.5 \mid Y = 1)$.

3.13 Consider a cantilever beam of span l m with a uniform load of w kg/m. The maximum shear force (S) and the maximum bending moment (M) at the fixed end are wl and $wl^2/2$, respectively. Consider l to be a constant and w to be a random variable with a mean of μ_w and a standard deviation of σ_w.

(a) Are S and M correlated? Calculate the covariance of S and M, that is, Cov(S,M).

(b) Calculate the correlation coefficient $\rho_{S,M}$.

(c) Using the results obtained in Parts (a) and (b), are S and M perfectly correlated?

Chapter 4

Commonly Used Probability Distributions

4.1 INTRODUCTORY COMMENTS

Modeling of uncertainties in both continuous and discrete random variables was discussed in Chapter 3. Any mathematical model satisfying the properties of PDF or PMF and CDF can be used to quantify uncertainties in a random variable. The procedures for selecting a particular distribution for a random variable and estimating its parameters to uniquely define the randomness will be discussed in Chapter 5. Many distributions are commonly used in the profession to calculate probability or reliability of events. Risk or reliability evaluation using these distributions is the subject of this chapter. Many computer programs and spreadsheets, such as MATLAB, Mathematica, EXCEL, QUATTRO PRO, etc., are used for probability calculations with many assumed distributions. Sometimes, these programs are used as "black boxes" without any understanding of the fundamentals of risk assessment. Conceptual and computational aspects of risk evaluation for an assumed distribution, without using any computer programs, are emphasized in this chapter.

4.2 CONTINUOUS RANDOM VARIABLES

For clarity of discussion, commonly used continuous and discrete random variables are discussed separately in the following sections.

4.2.1 Normal or Gaussian Distribution

One of the most commonly used distributions in engineering problems is the *normal* or *Gaussian distribution*. The PDF of the distribution can be expressed as

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right], \quad -\infty < x < +\infty \tag{4.1}$$

where the mean μ_X and the standard deviation σ_X are the two parameters of the distribution, usually estimated from the available data as discussed in Section 3.2 (and elaborated further in Chapter 5). The corresponding CDF can be expressed as

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right] dx. \tag{4.2}$$

The normal distribution is widely used and is denoted as $N(\mu, \sigma)$, indicating that it is a normal random variable with a mean and standard deviation of μ and σ, respectively. The PDF and CDF of a normal distribution with a mean of 100 and a standard deviation of 10 are shown in Figure 4.1. This distribution has many desirable features. It is applicable for any value of a random variable from $-\infty$ to $+\infty$. The distribution is symmetric about the mean, and the mean, median, and modal values are identical and can be estimated directly from the data. However, it is not simple to estimate the probability by integrating Equation 4.1. The problem can be addressed by transforming the original random variable X into a standard normal variable with zero mean and unit standard deviation, as

$$S = \frac{X-\mu_X}{\sigma_X}. \tag{4.3}$$

Using Equation 4.1 and the variable transformation technique (to be discussed in more detail in Chapter 6), we can express the PDF of S as

$$f_S(s) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}s^2\right], \quad -\infty < s < +\infty. \tag{4.4}$$

The corresponding CDF of S is

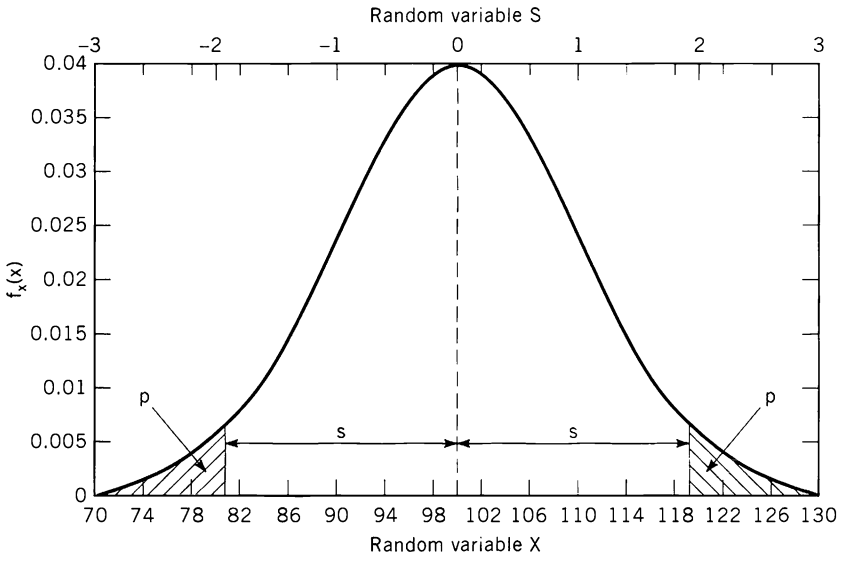
$$F_S(s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}s^2\right] ds. \tag{4.5}$$

The standard normal distribution is denoted as $N(0, 1)$, and its CDF is denoted as $\Phi(s)$, that is, $\Phi(s) = F_S(s)$, given by Equation 4.5. The CDF of the standard normal distribution is widely available in tabulated form, as shown in Appendix 1, or can be calculated using a standard subroutine available in many computer programs. Since the normal distribution is perfectly symmetrical, referring to Figure 4.1, we can show that

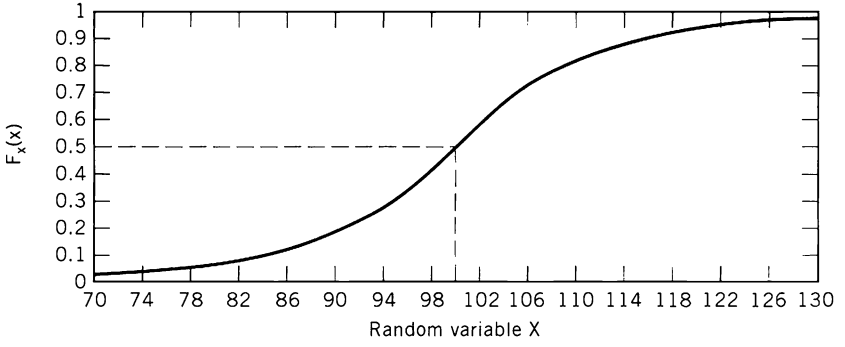
$$\Phi(-s) = 1.0 - \Phi(s) = p. \tag{4.6a}$$

or, when $p < 0.5$, we can show that

$$-s = \Phi^{-1}(p) = -\Phi^{-1}(1-p). \tag{4.6b}$$



(*a*) Normal PDF of X, N(100, 10) or S, N(0, 1)



(*b*) Normal CDF of X, N(100, 10)

Figure 4.1 PDF and CDF of Normal Random Variable

The table in Appendix 1 is only valid for positive values of the standard normal random variable. However, as discussed earlier, negative values are possible for a normal or standard random variable. The CDF of the standard normal variable can be evaluated for a negative value using Equation 4.6a.

EXAMPLE 4.1

Suppose $\Phi(-0.1)$ needs to be evaluated using the table in Appendix 1. Using the table, we find that

$$\Phi(0.1) = 0.53983.$$

Using Equation 4.6a, we can show that

$$\Phi(-0.1) = 1.0 - \Phi(0.1) = 1.0 - 0.53983 = 0.46017.$$

From Appendix 1, we find that $\Phi(0.0) = 0.5$. Thus, it can also be concluded that the CDF of the standard normal distribution evaluated at a negative value of S will be less than 0.5.

If a random variable X is $N(\mu_X, \sigma_X)$, then its probability of having a value between two limits a and b is

$$P(a < X \le b) = \frac{1}{\sigma_X \sqrt{2\pi}} \int_a^b \exp\left[-\frac{1}{2} \left(\frac{x - \mu_X}{\sigma_X} \right)^2 \right] dx. \tag{4.7}$$

Again, by variable transformation, Equation 4.7 can be rewritten in terms of the standard normal variable S as

$$P(a < X \le b) = \frac{1}{\sqrt{2\pi}} \int_{\frac{a-\mu_X}{\sigma_X}}^{\frac{b-\mu_X}{\sigma_X}} \exp\left[-\frac{1}{2} s^2 \right] ds$$

or

$$P(a < X \le b) = \Phi\left(\frac{b - \mu_X}{\sigma_X} \right) - \Phi\left(\frac{a - \mu_X}{\sigma_X} \right). \tag{4.8}$$

Equation 4.8 indicates that the probability of a normal random variable between two limits can be calculated easily using the table in Appendix 1.

EXAMPLE 4.2

To demonstrate the steps involved, consider the values of the Young's modulus given in Table 3.1. Assume that the randomness in E can be described by a normal random variable. Its mean and standard deviation were estimated in Section 3.2 as 29,576 ksi and 1,507 ksi, respectively. Using Equation 4.8, we can calculate the probability of E having a value between 28,000 ksi and 29,500 ksi as

$$\begin{aligned} P(28{,}000 < E \le 29{,}500) &= \Phi\left(\frac{29{,}500 - 29{,}576}{1{,}507} \right) - \Phi\left(\frac{28{,}000 - 29{,}576}{1{,}507} \right) \\ &= \Phi(-0.05) - \Phi(-1.05) = [1 - \Phi(0.05)] - [1 - \Phi(1.05)] \\ &= (1 - 0.51994) - (1.0 - 0.85314) = 0.33320. \end{aligned}$$

As stated earlier, the commonly used Young's modulus for steel is 29,000 ksi. The probability of the Young's modulus being less than the design value, that is, $-\infty < E \le 29{,}000$, can be calculated as

$$\begin{aligned} P(E \le 29{,}000) &= \Phi\left(\frac{29{,}000 - 29{,}576}{1{,}507} \right) - \Phi\left(\frac{-\infty - 29{,}576}{1{,}507} \right) \\ &= \Phi(-0.38) - \Phi(-\infty) \\ &= (1 - 0.64803) - 0.0 = 0.35197. \end{aligned}$$

This means that the design value of E is approximately the 35th percentile value for the data given in Table 3.1.

Similarly, the probability that Young's modulus will be at least 29,000 ksi can be calculated as

$$\begin{aligned} P(E \geq 29{,}000) &= P(29{,}000 < E \leq +\infty) \\ &= \Phi\left(\frac{\infty - 29{,}576}{1{,}507}\right) - \Phi\left(\frac{29{,}000 - 29{,}576}{1{,}507}\right) \\ &= \Phi(+\infty) - \Phi(-0.38) = 1 - 1 + \Phi(0.38) = 0.64803. \end{aligned}$$

Common sense suggests that the Young's modulus cannot be negative; thus, modeling it as a normal random variable may distort the physical aspects of the problem. This type of argument is often used against probabilistic analysis. The probability of E being less than or equal to zero can be calculated as

$$P(E \leq 0.0) = \Phi\left(\frac{0.0 - 29{,}576}{1{,}507}\right) - \Phi(-\infty) = \Phi(-19.63) \approx 0.0.$$

Obviously, there is virtually no impact on the uncertainty analysis of the Young's modulus. If the underlying distribution is normal and negative values are not possible, the available data should reflect the physical aspect of the random variable. This will be discussed further later.

The Young's modulus is a resistance-related random variable, whose design value is usually selected to be less than the mean value. Suppose its design value is selected to be the 10th percentile value. Denoting it as $e_{0.10}$, we can calculate the design value of the Young's modulus as

$$P(E \leq e_{0.10}) = 0.10$$

or

$$\Phi\left(\frac{e_{0.10} - 29{,}576}{1{,}507}\right) = 0.10$$

or

$$\left(\frac{e_{0.10} - 29{,}576}{1{,}507}\right) = \Phi^{-1}(0.10) = -\Phi^{-1}(0.90) = -1.28.$$

Thus,

$$e_{0.10} = 29{,}576 - 1.28 \times 1{,}507 = 27{,}647 \text{ ksi}.$$

If the 90th percentile value of the Young's modulus is desired, it can be calculated as

$$e_{0.90} = 29{,}576 + 1.28 \times 1{,}507 = 31{,}505 \text{ ksi}.$$

For the data under consideration, the design value of 29,000 ksi of the Young's modulus is approximately the 35th percentile value, as shown earlier. For a normal random variable, the probability associated with an event or the design value can be easily calculated using the simple procedure discussed above.

When a distribution like the normal distribution, which is valid from $-\infty$ to $+\infty$, lacks a physical interpretation to consider the practical aspects of the problem, it is common in the literature to consider the values of the random variable as belonging to

a range bounded by the mean plus and minus some standard deviation values. For a normal distribution, mean ± 3σ bounds are very common. If the data are limited to these lower and upper bounds, they will give a probability of 0.997 instead of 1.0, indicating that the error associated with the probability calculation is marginal. For the Young's modulus parameter under consideration, the lower and upper bounds will be 29,576 − 3 × 1,507 = 25,055 ksi and 29,576 + 3 × 1,507 = 34,097 ksi, respectively. Thus, considering the physical aspects of the parameter, values of the Young's modulus between 25,055 and 34,097 ksi will effectively give practical limits to the bounds and will include about 99.7% of the data. Table 3.1 indicates that there are no observations outside this range, validating this statement.

For a normal distribution, it is interesting to note that if mean ± 1σ bounds are used, about 68.3% of the data are included. If the bounds are increased to mean ± 2σ, about 95.4% of the data are included.

EXAMPLE 4.3

Suppose a steel cable has to carry a weight of 10 kips. Information on the strength of similar cables indicates that the strength of the cable, R, can be modeled by a normal random variable with a mean of 25 kips and a standard deviation of 5 kips. Calculate the probability that the cable will be unable to carry the weight, or the probability that the cable will break.

SOLUTION

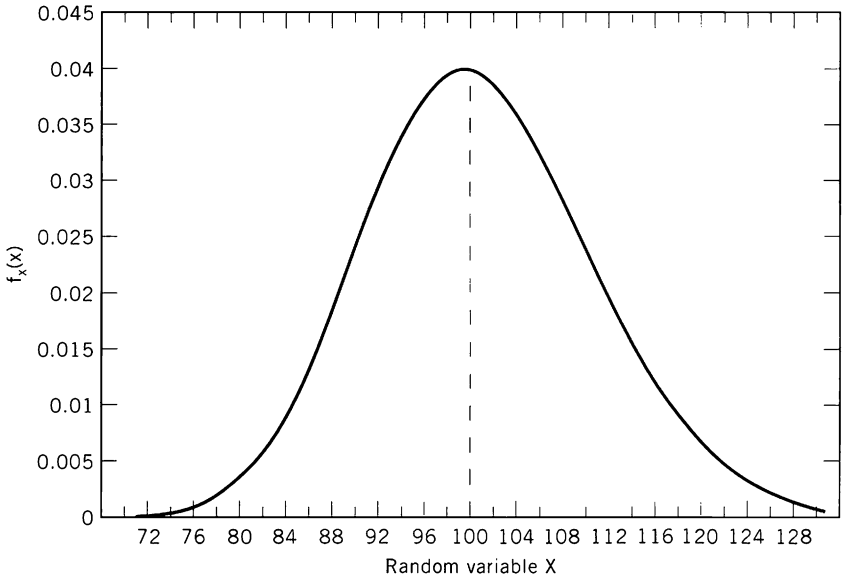
$$P(\text{the cable will break}) = P(\text{failure}) = P(R \le 10)$$
$$= \Phi\left(\frac{10-25}{5}\right) - \Phi\left(\frac{-\infty-25}{5}\right) = \Phi(-3) - \Phi(-\infty)$$
$$= 1 - \Phi(3) = 1 - 0.99865 = 0.00135.$$

4.2.2 Lognormal Random Variable

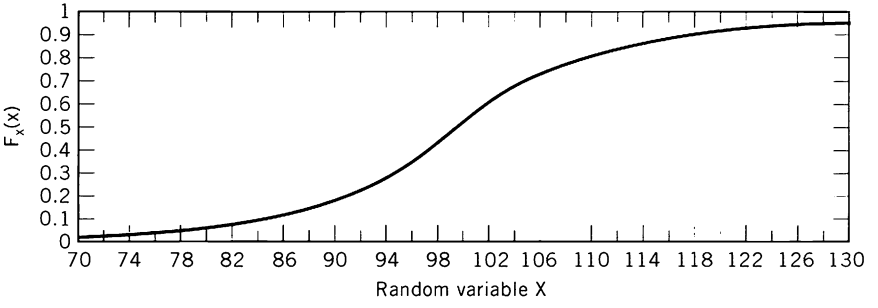
In many engineering problems, a random variable cannot have negative values due to the physical aspects of the problem. In this situation, modeling the variable as lognormal (i.e., considering the natural logarithm of the variable X) is more appropriate, automatically eliminating the possibility of negative values. If a random variable has a lognormal distribution, then its natural logarithm has a normal distribution. This is the meaning of the term *lognormal*. The PDF of a lognormal variable is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta_X x} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda_X}{\zeta_X}\right)^2\right], \quad 0 \le x < \infty$$

where λ_X and ζ_X are the two parameters of the lognormal distribution. The PDF of a typical lognormal distribution with a mean of 100 and a standard deviation of 10 is shown in Figure 4.2. The PDF values of both the normal and lognormal random variables with the same mean and standard deviation are plotted in Figure 4.3. The log-



(*a*) Lognormal PDF of X with mean = 100 and standard deviation = 10



(*b*) Lognormal CDF of X with mean = 100 and standard deviation = 10

Figure 4.2 PDF and CDF of Lognormal Random Variable

normal variable has values between zero and $+\infty$. Its PDF is unsymmetrical, and thus its mean, median, and modal values are expected to be different. Comparing Equations 4.1 and 4.9, we can observe some similarities between the normal and lognormal distributions. In fact, the two parameters of the lognormal distribution can be calculated from the information on the two parameters of the normal distribution, the mean (μ) and standard deviation (σ) of the sample population. It can be shown that

$$\lambda_X = E(\ln x) = \ln \mu_X - \frac{1}{2}\zeta_X^2 \tag{4.10}$$

and

$$\zeta_X^2 = \mathrm{Var}(\ln X) = \ln\left[1 + \left(\frac{\sigma_X}{\mu_X}\right)^2\right] = \ln\left(1 + \delta_X^2\right). \tag{4.11}$$

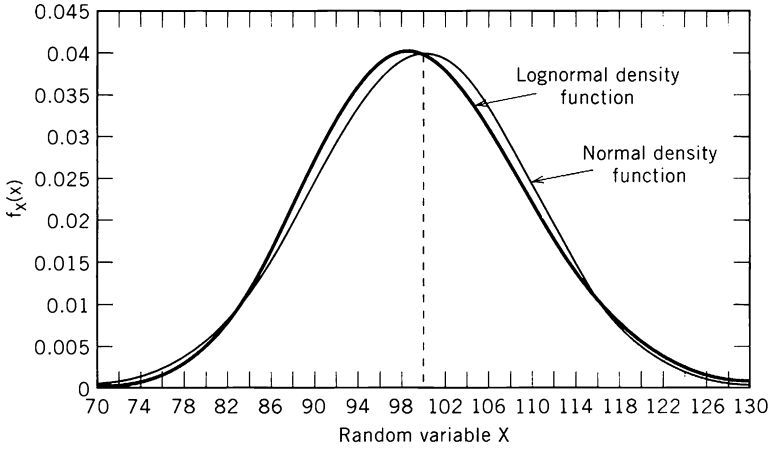


Figure 4.3 PDF of Normal and Lognormal Distributions with Mean =100 and Standard Deviation = 10

If the COV (i.e., $\delta_X = \sigma_X/\mu_X$) is not very large, for example less than 0.30, then $\zeta_X \approx \delta_X$, the COV of the random variable X.

To calculate the probability of an event where the underlying distribution of a random variable is lognormal, the procedures used for the normal variable are still applicable, except that for the lognormal case, the standard variable S will take the following form instead of Equation 4.3:

$$S = \frac{\ln X - \lambda_X}{\zeta_X}. \tag{4.12}$$

The probability of a lognormal random variable having a value between two limits a and b can be calculated as

$$\begin{aligned} P(a < X \le b) &= \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{\ln a - \lambda_X}{\zeta_X}\right)}^{\left(\frac{\ln b - \lambda_X}{\zeta_X}\right)} \exp\left(-\frac{1}{2}s^2\right) ds \\ &= \Phi\left(\frac{\ln b - \lambda_X}{\zeta_X}\right) - \Phi\left(\frac{\ln a - \lambda_X}{\zeta_X}\right). \end{aligned} \tag{4.13}$$

Thus, all required probabilities for the lognormal variable can be calculated from the CDF table developed for the standard normal variable, given in Appendix 1.

EXAMPLE 4.4

To demonstrate the calculation of probability for a lognormal variable, the same Young's modulus example with a mean of 29,576 ksi and a standard deviation of 1,507 ksi can be considered, except that now it is assumed that the Young's modulus is lognormally distributed. In this case,

$$\delta = \sigma / \mu = 1{,}507 / 29{,}576 = 0.051 \le 0.3.$$

Thus,

$$\zeta \approx \delta = 0.051$$

$$\lambda = \ln 29{,}576 - 0.5 \times 0.051^2 = 10.293.$$

The probability of E having a value between 28,000 ksi and 29,500 ksi can be calculated as

$$\begin{aligned} &P(28{,}000 < E \le 29{,}500) \\ &= \Phi\left(\frac{\ln 29{,}500 - 10.293}{0.051}\right) - \Phi\left(\frac{\ln 28{,}000 - 10.293}{0.051}\right) \\ &= \Phi(-0.017) - \Phi(-1.04) = (1 - 0.50678) - (1.0 - 0.85083) = 0.34405. \end{aligned}$$

Again, the probability of E being less than the design value of 29,000 ksi can be calculated as

$$\begin{aligned} P(E \le 29{,}000) &= \Phi\left(\frac{\ln 29{,}000 - 10.293}{0.051}\right) - \Phi(-\infty) = \Phi(-0.35) \\ &= 1.0 - 0.63683 = 0.36317. \end{aligned}$$

If E is modeled as a lognormal variable, the design value is about the 36th percentile value for the data given in Table 3.1.

If the design value for the Young's modulus is still the 10th percentile value, then it can be estimated as

$$\Phi\left(\frac{\ln e_{0.10} - 10.293}{0.051}\right) = 0.10 = \Phi(-1.28)$$

or

$$\ln e_{0.10} = 10.293 - 1.28 \times 0.051$$

or

$$e_{0.10} = 27{,}659 \text{ ksi}.$$

For the Young's modulus example under consideration, the results are similar for the normal and lognormal cases. This will be discussed further later.

Some of the important features of a lognormal variable can be summarized as follows:

(1) If X is a lognormal variable with parameters λ_X and ζ_X, then $\ln X$ is normal with a mean of λ_X and a standard deviation of ζ_X.

(2) $\lambda_X = \ln \mu_X - \frac{1}{2}\zeta_X^2$.

(3) $\zeta_X^2 = \ln(1 + \delta_X^2)$. When $\delta_X \le 0.3$, $\zeta_X \approx \delta_X$, the COV of X.

(4) Denoting x_m as the median of a lognormal variable X, we can show that $\lambda_X = \ln x_m$.

(5) $x_m = \dfrac{\mu_X}{\sqrt{1 + \delta_X^2}}$; that is, the median value of a lognormal variable is always less than the mean value.

4.2.3 Beta Distribution

The *beta distribution* is a very flexible and useful distribution and can be used when a random variable is known to be bounded by two limits, a and b. The normal distribution is valid between $-\infty$ and $+\infty$, and the lognormal distribution is valid between 0 and $+\infty$. Many random variables of engineering significance may be bounded by two limits, and the beta distribution could be quite appropriate. The PDF of a beta distribution is represented as

$$f_X(x) = \frac{1}{B(q,\ r)} \frac{(x-a)^{q-1}(b-x)^{r-1}}{(b-a)^{q+r-1}}, \quad a \le x \le b$$
$$= 0, \quad \text{elsewhere.} \tag{4.14}$$

where q and r are the parameters of the distribution and $B(q, r)$ is the *beta function*. The parameters q and r can be estimated from the mean and standard deviation of the available data using the following relationships:

$$E(X) = a + \frac{q}{q+r}(b-a) \tag{4.15}$$

and

$$\text{Var}(X) = \frac{qr}{(q+r)^2(q+r+1)}(b-a)^2. \tag{4.16}$$

If the upper and lower limits and the mean and variance of a random variable are known, the corresponding q and r parameters of the beta distribution can be estimated using Equations 4.15 and 4.16.

The beta function in Equation 4.14 can be shown to be

$$B(q,\ r) = \int_0^1 x^{q-1}(1-x)^{r-1}\,dx. \tag{4.17}$$

The beta function can also be calculated as

$$B(q,\ r) = \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r)} \tag{4.18}$$

where $\Gamma(\ )$ is the gamma function. Procedures to calculate gamma function are given in Appendix 2.

If the lower limit a is 0 and the upper limit b is 1, Equation 4.14 takes the following form:

$$f_X(x) = \frac{1}{B(q,\ r)} x^{q-1}(1-x)^{r-1}, \quad 0 \le x \le 1$$
$$= 0, \quad \text{elsewhere.} \tag{4.19}$$

Equation 4.19 is known as the *standard beta distribution*. Its PDFs for several values of q and r are shown in Figure 4.4. When q and r are both equal to one, the beta distribution becomes a *uniform distribution*.

Once the PDF of a beta distribution is defined, the probability of any event can be estimated by numerically integrating the area under the PDF corresponding to the upper and lower limits. The probability can also be found using tables for the standard beta distribution similar to the standard normal table given in Appendix 1.

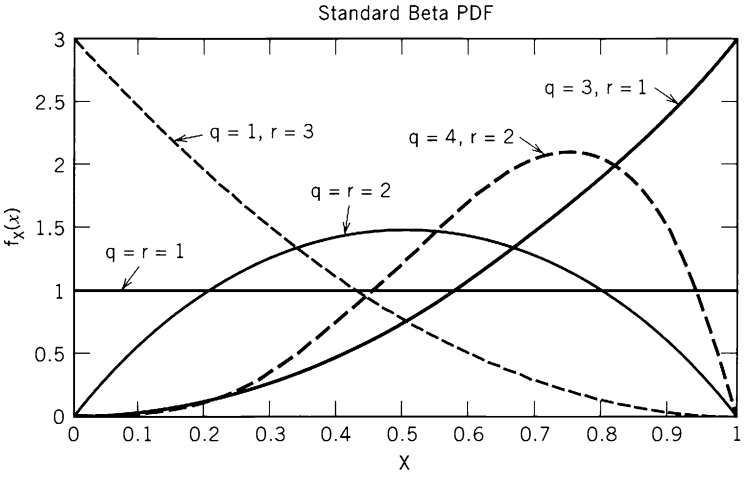


Figure 4.4 Standard Beta PDF

EXAMPLE 4.5

The daily maximum temperature in June in Tucson, Arizona, varies between 80°F and 110°F. Using 100 years of data, we can estimate that the average daily maximum temperature in June is 95°F and the corresponding standard deviation is 10°F. Assuming that the daily maximum temperature can be modeled by a beta distribution, what is the probability that on any given day in June, the daily maximum temperature will exceed 100°F?

It is clear that modeling the temperature by a normal or lognormal distribution may not be appropriate. The beta distribution may be a reasonable alternative. The following information can be extracted from the available data. For the beta distribution under consideration, $a = 80$, $b = 110$. Also, using Equations 4.15 and 4.16, we can show that

$$80 + \frac{q}{q+r}(110-80) = 95$$

and

$$\frac{qr}{(q+r)^2(q+r+1)}(110-80)^2 = 10^2.$$

In this particular case, $q = r = 5/8$, and using Equation 4.18,

$$B\left(\frac{5}{8},\frac{5}{8}\right) = \frac{\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{5}{8}+\frac{5}{8}\right)}.$$

Gamma functions can be evaluated using Appendix 2, where $\Gamma(5/8)$ is shown to be 1.434519178. Using the first equation in Section 2a of Appendix 2, we can calculate $\Gamma(5/8 + 5/8)$ or $\Gamma(5/4)$ as

$$\Gamma(5/4) = \Gamma(1+1/4) = 1 - 0.5748646\times(1/4) + 0.9512363\times(1/4)^2 - 0.6998588\times(1/4)^3 + 0.4245549\times(1/4)^4$$

$$
\begin{aligned}
&- 0.1010678 \times (1/4)^5 \\
&= 0.906360543.
\end{aligned}
$$

Thus,

$$
B\left(\frac{5}{8}, \frac{5}{8}\right) = \frac{(1.434519178)^2}{0.906360453} = 2.270.
$$

Computer programs like EXCEL, QUATTRO PRO, and MATLAB can also be used to calculate the gamma functions. To demonstrate how to use Appendix 2, we have evaluated the gamma functions up to nine decimal place accuracy. However, considering the practical aspect of the problem, the beta function in Equation 4.14 or 4.19 need not be calculated with similar accuracy.

Once the beta function is evaluated, the corresponding PDF of the beta distribution of the daily maximum temperature, T, is given by Equation 4.14 and can be shown to be

$$
\begin{aligned}
f_T(t) &= \frac{1}{2.270} \frac{(t-80)^{\frac{5}{8}-1}(110-t)^{\frac{5}{8}-1}}{(110-80)^{\frac{5}{8}+\frac{5}{8}-1}} \\
&= \frac{1}{5.313}(t-80)^{-\frac{3}{8}}(110-t)^{-\frac{3}{8}}, \quad 80 \le t \le 110.
\end{aligned}
$$

Thus, if we numerically integrate the above PDF from 100°F to 110°F, the probability that on any given day in June, the daily maximum temperature in Tucson will exceed 100°F can be calculated as

$$
P(T > 100) = \int_{100}^{110} \frac{1}{5.313}(t-80)^{-\frac{3}{8}}(110-t)^{-\frac{3}{8}} dt = 0.3716.
$$

This example demonstrates that depending upon the mathematical form of the PDF, it may be necessary to numerically integrate the area under the PDF between the upper and lower limits to estimate the corresponding probability.

Another example of the application of the beta distribution is given in Section 4.4.

4.3 DISCRETE RANDOM VARIABLES

4.3.1 Binomial Distribution

In many engineering applications, events consisting of repeated trials can be formulated in terms of occurrence or nonoccurrence, success or failure, good or bad, and so forth. Only two outcomes are possible, representing the behavior of a discrete random variable. In addition, if the events satisfy the additional requirements of a *Bernoulli sequence*, that is, if they are statistically independent and the probability of occurrence or nonoccurrence of events remains constant, they can be mathematically represented by the *binomial distribution*. If the probability of occurrence of an event in each trial is p and the probability of nonoccurrence is $(1 - p)$, then the probability

of x occurrences out of a total of n trials can be described by the PMF of a binomial distribution as

$$P(X = x, n \mid p) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, 2, \ldots, n \tag{4.20}$$

where p, the probability of occurrence in each trial, is the parameter of the distribution, and $\binom{n}{x} = n!/[x!(n-x)!]$ is the *binomial coefficient*, indicating the number of ways that x occurrences out of a total of n trials are possible. Note that X, the number of occurrences, is a discrete random variable, since it can only take integer values.

EXAMPLE 4.6

Suppose the probability of failure of a structure due to earthquakes is estimated as 10^{-5} per year. Assuming that the design life of the structure is 50 years and the probability of failure in each year remains constant and independent during its lifetime, then the probability of no failure can be estimated using the binomial distribution as

$$\begin{aligned} P(\text{no failure in 50 years}) &= P(X = 0, 50 \mid 10^{-5}) \\ &= \binom{50}{0} (10^{-5})^0 (1 - 10^{-5})^{50-0} = \frac{50!}{0!(50-0)!} (1 - 10^{-5})^{50} \\ &\approx 1 - 50 \times 10^{-5} = 0.99950 \\ P(\text{failure in 50 years}) &= 1 - P(\text{no failure in 50 years}) \\ &= 1 - 0.99950 = 0.00050. \end{aligned}$$

EXAMPLE 4.7

The probabilistic characteristics of the car problem considered in Example 3.2 can be described by the binomial distribution. Since three cars are involved, $n = 3$. Also, the probability of each car being good is 0.9, or $p = 0.9$. The binomial coefficients when $X = 0$, 1, 2, and 3 can be shown to be 1, 3, 3, and 1, respectively, indicating the total number of sample points in each event. Thus, using Equation 4.20, we can calculate the PMF of X when it is 0, 1, 2, and 3 by taking the product of the probability of occurrence of one sample point in any event and the corresponding binomial coefficients, as shown in Example 3.2. The PMFs of X when it is 0, 1, 2, and 3 are shown in Figure 3.4. Once the PMFs are available, the corresponding CDFs of X can be easily calculated, as shown in Figure 3.4.

EXAMPLE 4.8

The drainage system of a city has been designed for a rainfall intensity that will be exceeded on an average once in 50 years. What is the probability that the city will be flooded at most 2 out of 10 years?

SOLUTION

Since the possible outcomes in each year consist of flooding or nonflooding, the problem can be modeled as a binomial distribution. In this case, the parameter p (the probability of flooding in one year) is 1/50 = 0.02. Thus,

$$\begin{aligned} P(\text{flooding in 2 out of 10 years}) &= P(X = 2,10 \mid 0.02) \\ &= \binom{10}{2}(0.02)^2(1-0.02)^{10-2} \\ &= \frac{10!}{2!(10-2)!}(0.02)^2(0.98)^8 = 0.015. \end{aligned}$$

The probability of flooding in at most 2 years out of 10 years can be calculated as

$$\begin{aligned} &P(X = 0,10 \mid 0.02) + P(X = 1,10 \mid 0.02) + P(X = 2,10 \mid 0.02) \\ &= \binom{10}{0}(0.02)^0(0.98)^{10} + \binom{10}{1}(0.02)^1(0.98)^9 + \binom{10}{2}(0.02)^2(0.98)^8 \\ &= 0.817 + 0.167 + 0.015 = 0.999. \end{aligned}$$

It is interesting to note that there is a probability of 0.817 that the city will not be flooded in any of 10 years.

Similarly, the probability of no flood in 50 years can be shown to be $0.98^{50} = 0.364$, although on average the city is expected to be flooded once in 50 years. This will be elaborated upon in Section 4.3.3.

4.3.2 Geometric Distribution

The first occurrence time of an event is of great interest in engineering. Information on the first time the design wind speed will be exceeded in an area or the first time a structure will be damaged by earthquakes is important. If the events occur in a Bernoulli sequence and p is the probability of occurrence in each trial, then the probability that the event will occur for the first time at the ith trial—which implies that there was no occurrence in the previous $(t-1)$ trials—is given by the *geometric distribution* as

$$P(T = t) = p(1-p)^{t-1} \quad t = 1,2,\ldots. \tag{4.21}$$

Considering Example 4.6, the probability of the failure of the structure in the 10th year can be calculated as

$$P(T = 10) = 10^{-5}(1-10^{-5})^{10-1} = 9.999 \times 10^{-6}.$$

4.3.3 Return Period

The design wind speed, rainfall, or flood level at a particular location is usually expressed in terms of *return period*. Suppose an event occurs in a Bernoulli sequence, and it occurs for the first time after T_1 years. It occurs again T_2 years after the first occurrence, and again T_3 years after the second occurrence, and so on. The *recurrence time*, the time between two consecutive occurrences of the same event, must follow the

probabilistic characteristics of the first occurrence, that is, the geometric distribution whose PMF is given by Equation 4.21. Thus, Equation 3.19 can be used to calculate the mean recurrence time, also known as the return period:

$$\begin{aligned}\text{return period } T = E(T) &= \sum_{t=1}^{\infty} t p_T(t) = \sum_{t=1}^{\infty} t p(1-p)^{t-1} \\ &= p\left[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \ldots\right]\end{aligned} \tag{4.22}$$

The terms in square brackets in Equation 4.22 represent an infinite series and can be shown to be $1/p^2$. Thus, Equation 4.22 can be simplified to

$$\text{return period } T = p \times \frac{1}{p^2} = \frac{1}{p}. \tag{4.23}$$

Equation 4.23 states that if the design wind speed corresponds to a 50-year return period, then the probability in each year that the design wind speed will be exceeded is 1/50 = 0.02; on average, the design wind speed will be exceeded once every 50 years. It must be noted that the design wind speed can be exceeded several times or not at all within the return period but on average will be exceeded once in 50 years.

In Example 4.8, the design flood level is considered to have a return period of 50 years, indicating that on average there will be a flood once every 50 years. However, there is a probability of 0.364 that no flood will occur in the next 50 years.

4.3.4 Poisson Distribution

Another important distribution used frequently in engineering to evaluate the risk of damage is the *Poisson distribution*. Defects can occur at any location along the length of welds. A tornado can strike a structure at any time during its lifetime. An accident can occur at any location along a highway. These events can occur at any point in time or space. If they need to be modeled in a Bernoulli sequence, that is, occurrence or nonoccurrence at a given time or space, the total space or time needs to be subdivided into very small intervals so that only one occurrence is possible in an interval. Suppose that the mean occurrence rate of tornadoes at a location is v times a year. Thus, over a period of t years, tornadoes will occur an average of (vt) times. If the time period t is divided into n intervals, then the probability of tornado occurrence in each interval will be $(vt)/n$. Modeling x occurrences in time t in a Bernoulli sequence as n approaches infinity will lead to the Poisson distribution, which can be expressed as

$$\begin{aligned}P(x \text{ occurences in time } t) &= \lim_{n\to\infty}\binom{n}{x}\left(\frac{vt}{n}\right)^x\left(1-\frac{vt}{n}\right)^{n-x} = \lim_{n\to\infty}\left[\frac{n!}{x!(n-x)!}\right]\left(\frac{vt}{n}\right)^x\left(1-\frac{vt}{n}\right)^{n-x} \\ &= \lim_{n\to\infty}\left[\frac{n}{n}\bullet\frac{(n-1)}{n}\cdots\frac{(n-x+1)}{n}\bullet\frac{(vt)^x}{x!}\left(1-\frac{vt}{n}\right)^n\left(1-\frac{vt}{n}\right)^{-x}\right] \\ &= \lim_{n\to\infty}\left[\frac{(vt)^x}{x!}\left(1-\frac{vt}{n}\right)^n\right].\end{aligned}$$

Taking the limit of this equation, and knowing that

$$\lim_{n\to\infty}\left(1-\frac{vt}{n}\right)^n = 1 - vt + \frac{(vt)^2}{2!} - \frac{(vt)^3}{3!} + \ldots = e^{-vt},$$

we can show that

$$P(x \text{ occurences in time } t) = \frac{(vt)^x}{x!}e^{-vt}. \tag{4.24}$$

Equation 4.24 represents the PMF of the Poisson distribution.

EXAMPLE 4.9

From records of the past 50 years, it is observed that tornadoes occur in a particular area an average of two times a year. In this case, v = 2/year. The probability of no tornadoes in the next year (i.e., $x = 0$, and $t = 1$ year) can be calculated as

$$P(\text{no tornado next year}) = \frac{(2\times 1)^0 \bullet e^{-2\times 1}}{0!} = 0.135$$

$$P(\text{exactly 2 tornadoes next year}) = \frac{(2\times 1)^2 \bullet e^{-2\times 1}}{2!} = 0.271$$

$$P(\text{no tornado in next 50 years}) = \frac{(2\times 50)^0 \bullet e^{-2\times 50}}{0!} = 3.72\times 10^{-44}.$$

The results indicate that, for a tornado-prone area where an average of two tornadoes per year are expected, the probability of no tornadoes in the 50-year design life is very close to zero, essentially an impossible event. In other words, the probability of at least one tornado in the next 50 years will be very close to 1 ($1 - 3.72 \times 10^{-44}$), indicating that it is almost a certainty.

EXAMPLE 4.10

For a large construction project, the contractor estimates that the average rate of on-the-job accidents is three times per year. From past experience, the contractor also estimates that the cost incurred for each accident may be modeled as a lognormal random variable with a median of \$6,000 and COV of 20%. The cost of each accident can be assumed to be statistically independent.

(a) What is the probability that there will be no accident in the first month of construction?

(b) What is the probability that only 1 out of the first 3 months of construction is free of accidents?

(c) What is the probability that an accident will incur a loss exceeding \$4,000?

(d) What is the probability that none of the accidents in a month will cost more than \$4,000?

SOLUTION

(a) For the Poisson distribution, $v = 3$ times per year $= 3/12 = 1/4$ time per month, $t = 1$ month, $vt = 1/4 \cdot 1 = 1/4$, and no accident means $x = 0$. Thus, using Equation 4.24,

$$P(\text{no accident in the first month}) = \frac{e^{-\frac{1}{4}} \cdot \left(\frac{1}{4}\right)^0}{0!} = e^{-\frac{1}{4}} = 0.7788.$$

(b) The binomial distribution needs to be used in this case. Thus, using Equation 4.20,

$$P(X = 1, 3 \mid 0.7788) = \binom{3}{1}(0.7788)^1(1 - 0.7788)^{3-1} = 0.1143.$$

(c) The cost incurred for each accident is modeled as a lognormal distribution. In this case, using Equation 4.13, $\delta \approx \zeta = 0.2$, and $\lambda = \ln(\text{median}) = \ln 6{,}000 = 8.70$. Thus,

$$P(\text{cost of an accident} > \$4{,}000) = \Phi\left(\frac{\ln \infty - 8.7}{0.20}\right) - \Phi\left(\frac{\ln 4{,}000 - 8.7}{0.20}\right)$$
$$= 1 - \Phi(-2.027) = 1 - 0.0213 = 0.9787.$$

(d) This can be solved by considering that there could be n number of accidents in a month; n could be any number, and no accident should exceed a cost of \$4,000. From Part (c),

$$P(\text{an accident will cost less than } \$4{,}000) = 1 - 0.9787 = 0.0213.$$

Thus,

$$= \sum_{n=0}^{\infty} P(\text{cost of an accident} \le \$4{,}000 \mid X = n) P(X = n)$$

$$= \sum_{n=0}^{\infty} (0.0213)^n \cdot \frac{e^{-\frac{1}{4}}\left(\frac{1}{4}\right)^n}{n!}$$

$$= e^{-\frac{1}{4}}\left[\sum_{n=0}^{\infty} \frac{\left(0.0213 \cdot \frac{1}{4}\right)^n}{n!}\right] = e^{-\frac{1}{4}} \cdot e^{\left(0.0213 \cdot \frac{1}{4}\right)} = e^{-0.2447} = 0.78296.$$

Note that the infinite series in the third bracket in this equation is an exponential series; that is, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$. If is not obvious, then the first few terms of the series (perhaps 3 or 4) can be considered to calculate the probability.

EXAMPLE 4.11

The safety of a building in an earthquake-prone area is under consideration. The past 100 years of data indicate that there were four strong earthquakes in the area. Also, a

detailed evaluation indicates that during a strong earthquake, the probability that the building will suffer damage is 0.10. Assume that damage events for different earthquakes are statistically independent.

(a) What is the probability that there will be no strong earthquake in the area in 50 years, the service life of the building?
(b) What is the probability that there will be only two strong earthquakes in 50 years?
(c) What is the probability that the building will suffer damage due to strong earthquakes in 50 years?

SOLUTION

(a) In this case, the average rate of strong earthquake occurrences, v, is 4/100 = 0.04 per year. Thus, $vt = 0.04 \times 50 = 2$.

$$P(\text{no strong earthquake in 50 years}) = P(X = 0)$$
$$= \frac{e^{-2} \times (2)^0}{0!} = 0.13534$$

(b) $P(\text{two strong earthquakes in 50 years}) = P(X = 2)$

$$= \frac{e^{-2} \times (2)^2}{2!} = 0.27067$$

(c) Let D denote the event that the building will suffer earthquake damage in 50 years. Then,

$$P(D) = 1.0 - P(\overline{D})$$
$$= 1.0 - \sum_{i=0}^{\infty} P(\overline{D} \mid X = i) P(X = i)$$
$$= 1.0 - \sum_{n=0}^{\infty} (1.0 - 0.1)^n \frac{e^{-2} \times (2)^n}{n!}$$
$$= 1.0 - e^{-2} \sum_{n=0}^{\infty} \frac{(0.9 \times 2)^n}{n!} = 1.0 - e^{-2} e^{1.8} = 1.0 - e^{-0.2} = 0.18127.$$

Again, the infinite series in this equation is an exponential series, as discussed in the previous example.

4.3.5 Exponential Distribution

If events occur according to a Poisson process, then the time T before the first occurrence of the event—implying no occurrence ($x = 0$) in time t—can be represented by the *exponential distribution*. It can be shown that

$$P(T > t) = \frac{e^{-vt}(vt)^0}{0!} = e^{-vt}. \tag{4.25}$$

Thus, the CDF of T can be shown to be

$$F_T(t) = P(T \le t) = 1 - e^{-\nu t} \tag{4.26}$$

and the corresponding PDF of the exponential distribution is

$$f_T(t) = \frac{dF_T(t)}{dt} = \nu e^{-\nu t}, \quad t \ge 0. \tag{4.27}$$

The mean value of T can be shown to be $1/\nu$. Simply stated, the mean of the first occurrence time or the recurrence time, or simply the return period for the Poisson model, is $1/\nu$. It is interesting to note that when events occur in a Bernoulli sequence, the return period is $1/p$ (Equation 4.23 in Section 4.3.3). It can be shown that for events with a small occurrence rate ν, the return periods according to the Bernoulli sequence and the Poisson model are approximately the same.

EXAMPLE 4.12

In the previous example (Example 4.11), strong earthquakes in an area are assumed to occur according to the Poisson distribution with the average rate of occurrences ν = 0.04 per year. The recurrence time T, the time between two consecutive occurrences of strong earthquakes, can be modeled by an exponential distribution as

$$f_T(t) = 0.04e^{-0.04t}, \quad t \ge 0.$$

The mean recurrence time or the return period of strong earthquakes can be shown to be

$$\text{return period } T = \int_0^\infty t \times 0.04 \times e^{-0.04t} dt = \frac{1}{0.04} = 25 \text{ years.}$$

The probability of no strong earthquakes in 50 years can be calculated as

$$P(T > 50) = \int_{50}^\infty 0.04 \times e^{-0.04t} dt = e^{-0.04 \times 50} = 0.13534.$$

The same result was obtained when the occurrences of strong earthquakes were assumed to follow a Poisson distribution, as shown in the previous section.

EXAMPLE 4.13

In earthquake engineering, the PDF for earthquake intensities, for example in Modified Mercalli (MM) scale, is sometimes modeled by an exponential distribution. The parameter ν is determined from local seismicity records.

In earthquake-resistant design of nuclear power plants, unserviceability and collapse due to earthquakes are the two most important concerns for engineers. The corresponding earthquake intensities are known in the profession as the operating basis earthquake (OBE) and the safe shutdown earthquake (SSE), respectively. One way to design for these incidents is to choose a design intensity x_i such that the probability that this intensity level is exceeded, that is, $P(X > x_i) = p$, is small. Since the collapse of a nuclear

power plant presents a great hazard to the public, the chance of its occurrence should be extremely small. Suppose a design intensity x_1 corresponding to a risk level of 10^{-3} is chosen for the OBE, and x_2 corresponding to a risk level of 10^{-6} is chosen for the SSE.

(a) Determine x_2 (SSE intensity) in terms of x_1 (OBE intensity).

(b) If power plant service is interrupted during an earthquake, what is the probability that the plant will collapse?

SOLUTION

(a) Using Equation 4.25, we can summarize the information in the problem as

$$P(X > x_1) = 10^{-3} \quad \text{and} \quad P(X > x_2) = 10^{-6}.$$

Thus,

$$e^{-\nu x_1} = 10^{-3} \quad \text{and} \quad e^{-\nu x_2} = 10^{-6}.$$

By simplifying,

$$x_1 = \frac{6.908}{\nu} \quad \text{and} \quad x_2 = \frac{13.816}{\nu}.$$

Or, $x_2 = 2.0\, x_1$. In this particular example, the SSE intensity is twice the OBE intensity.

(b) P(plant will collapse | service has been interrupted)

$$= P(X > x_2 \mid X > x_1) = \frac{P[(X > x_2)(X > x_1)]}{P(X > x_1)}$$

$$= \frac{P(X > x_2)}{P(X > x_1)} = \frac{10^{-6}}{10^{-3}} = 10^{-3}$$

EXAMPLE 4.14

The rate of oxygen consumption, D, caused by wastes discharged into a river, expressed in terms of biological oxygen demand (BOD), depends on the remaining BOD concentration. Suppose D can be described by an exponential distribution.

(a) If the mean value of D is found to be 6 mg/m^{3}d, define its PDF.

(b) What is the probability that D will be less than or equal to 4 mg/m^{3}d?

SOLUTION

(a) One attractive property of an exponential distribution is that its parameter ν in Equation 4.27 is the reciprocal of the mean or standard deviation. It is given in Table 5.6 in Chapter 5. Thus, for the problem under consideration, $\nu = 1/6$. The PDF of D can be shown to be

$$f_D(d) = \frac{1}{6} e^{-\frac{d}{6}}.$$

(b)

$$P(D \le 4) = \int_0^4 \frac{1}{6} e^{-\frac{d}{6}} dd = \left[-e^{-\frac{d}{6}} \right]_0^4 = -0.51342 + 1.0 = 0.48658.$$

4.4 A COMBINATION OF CONTINUOUS AND DISCRETE RANDOM VARIABLES: HYPERGEOMETRIC AND HYPERBINOMIAL DISTRIBUTIONS

The concept of the binomial distribution, discussed in Section 4.3.1, appears to be very simple; however, its applications could be limited in real engineering problems, since a prior estimate of p must be available. This is rarely the case. Moreover, the information on p must be extracted from limited samples, making it unreliable. The use of the binomial distribution with the uncertainty-filled parameter p makes its results questionable. The problem can be stated in the following way.

Suppose the reliability of welds in a structure needs to be estimated. For mathematical representation, the total number of discrete welds can be represented as N. The number of good or bad welds and the corresponding success or failure rate of the welds are not known in advance. Generally, the success or failure rate is determined by conducting nondestructive experiments. However, it is not practical or economical to inspect all N welds. Suppose that a sample of n welds is selected at random for inspection. Considering the cost of inspections, n should be a small fraction of N. Assume that m out of n inspected welds are found to be good, giving a success rate of $p = m/n$. However, the value of p can be questioned since it is based on limited sample information. The total number of good welds in the structure is expected to be $N \cdot p$, and the remaining $(N - N \cdot p)$ are bad. If a sample of n welds is taken at random from the structure, the probability of x good welds in the sample is given by the *hypergeometric distribution* (Ang and Tang, 1975) as

$$P(X = x) = \frac{\binom{N \cdot p}{x}\binom{N - N \cdot p}{n - x}}{\binom{N}{n}} \tag{4.28}$$

in which $\binom{N}{n}$ is the binomial coefficient defined earlier.

If n is small relative to N (about 5–10% of N in real problems), we can show that the hypergeometric distribution can be approximated by the binomial distribution, and its PMF can be represented by Equation 4.20. Equation 4.20 is appropriate when the value of p is known precisely. When p is not known but must be estimated, then such a binomial distribution assumes more precision than actually exists, making the situation appear better than it really is. Haldar (1982) showed that the probabilistic characteristics of p can be described by the beta distribution discussed in Section 4.2.3.

Furthermore, since p can only have values between 0 and 1, it can be represented by the standard beta distribution with parameters q and r as in Equation 4.19. The parameters q and r can be estimated from the inspection outcomes (i.e., a total of n welds are inspected and m of them are found to be good), as

$$q = m + 1 \tag{4.29a}$$

and

$$r = n - m + 1. \tag{4.29b}$$

The corresponding PDF of p can be represented by the standard beta distribution (Equation 4.19) as

$$f_P(p) = \frac{1}{B(q,r)} p^m (1-p)^{n-m}.$$

Using Equation 4.18,

$$B(q,r) = \frac{\Gamma(m+1)\Gamma(n-m+1)}{\Gamma(n+2)} = \frac{m!(n-m)!}{(n+1)!}.$$

The PDF of p can be shown to be

$$\begin{aligned} f_P(p) &= \frac{(n+1)!}{m!(n-m)!} p^m (1-p)^{n-m}, \quad 0 \le p \le 1.0 \\ &= 0, \quad \text{elsewhere.} \end{aligned} \tag{4.30}$$

Thus, while the expected value of p is m/n, other values of p between 0 and 1 are possible and they cannot be ignored in any subsequent inferences.

A joint or a structural element can contain h welds. To obtain the probability of k good welds out of h, based on the sampling information that n welds have been tested and m of them were found to be good, Equations 4.20 and 4.30 can be combined to obtain the unconditional distribution as

$$P(k \text{ of } h \mid m \text{ of } n) = \int_0^1 P(k \text{ of } h \mid p) f_P(p \mid m \text{ of } n) dp \tag{4.31a}$$

$$= \int_0^1 \frac{h!}{k!(h-k)!} p^k (1-p)^{h-k} \frac{(n+1)!}{m!(n-m)!} p^m (1-p)^{n-m} dp. \tag{4.31b}$$

Using combinational notation, we can rearrange Equation 4.31b to obtain the *hyperbinomial distribution* as

$$P(k \text{ of } h \mid m \text{ of } n) = \frac{\binom{m+k}{m}\binom{n+h-m-k}{h-k}}{\binom{n+h+1}{h}}, \tag{4.32}$$

$$\text{for } k = 0,1,2,\ldots,h, \text{ and } m \le n.$$

The CDF of the hyperbinomial distribution can be obtained from the PMF given by Equation 4.32. With this information, the number of good welds k out of a total h welds installed in a particular joint or structural element can be estimated with some predetermined confidence level based on the sampling information. A 90% confidence level may be reasonable. A computer program can be used for this purpose. Suppose that at least k welds are desired to be good with a 90% confidence level; then,

$$P(K \geq k) = 1.0 - P(K < k) = 0.90. \tag{4.33}$$

Equation 4.33 indicates that the CDF instead of the PMF of the hyperbinomial distribution is needed to solve the problem. The CDF of the hyperbinomial distribution can be easily obtained by taking the sum of the PMFs given by Equation 4.32.

EXAMPLE 4.15

A real application of the hyperbinomial distribution was given by Haldar (1982). The problem can be simplified as follows.

A total of 26,500 welds are installed in a building. The quality of the installed welds is suspect and reinspection of them is necessary. However, for many reasons, only 2,350 are available for inspection and 1,902 of these are found to be good. Suppose a joint in the building contains 10 welds. For repair purposes, it is important to find out how many of them are good with a predetermined confidence level, such as 90%.

SOLUTION

For the problem under consideration, $N = 26{,}500$, $n = 2{,}350$, $m = 1{,}902$, $p = 1{,}902/2{,}350 = 0.809$, $h = 10$, and k needs to be estimated with a 90% confidence level. The number of good welds and the corresponding CDFs of the binomial and hyperbinomial distributions are given in Table 4.1.

Several important observations can be made from this example. The average number of good welds out of a total of 10 can be shown to be $h \bullet p = 0.809 \times 10 \approx 8$. As the confidence level increases, a smaller number out of a total of 10 welds are expected to be good. Using Equation 4.33, we can observe from Table 4.1 that for the example under consideration, at least 6 out of 10 will be expected to be good with a 97% confidence level, whereas at least 9 out of 10 will be expected to be good with a confidence level of only 40%.

Table 4.1 Binomial and Hyperbinomial Distributions

	Binomial Distribution	Hyperbinomial Distribution	
Number of good welds, k (1)	$P(K \leq k, 10 \mid p = 0.809)$ (2)	$P(K \leq k \mid h = 10, n = 2{,}350, m = 1{,}902)$ (3)	$P(K \leq k \mid h = 10, n = 235, m = 190)$ (4)
0	6.340 E –08	6.966 E –08	1.574 E –07
1	2.755 E –06	2.964 E –06	5.623 E –06
2	5.418 E –05	5.723 E –05	9.308 E –05
3	6.364 E –04	6.618 E –04	9.424 E –04
4	4.962 E –03	5.094 E –03	6.487 E –03
5	2.700 E –02	2.743 E –02	3.193 E –02
6	1.050 E –01	1.058 E –01	1.150 E –01
7	2.941 E –01	2.951 E –01	3.060 E –01
8	5.953 E –01	5.956 E –01	6.013 E –01
9	8.794 E –01	8.792 E –01	8.792 E –01
10	1.000 E 00	1.000 E 00	1.000 E 00

The success rate parameter p is basically a random variable. A precise value of p can be had only when the entire sample space is tested, which is not practical in most cases. Thus, the binomial distribution with a point estimate of p from limited sample information is inappropriate. The CDFs for the binomial and hyperbinomial distributions when $p = 1{,}902/2{,}350 = 0.809$ are shown in Columns 2 and 3 of Table 4.1, respectively. The CDF values are not identical. Considerable differences are expected between the two CDF values, although the differences are small for the example considered here. Generally, the PMFs of the hyperbinomial distribution are more diffuse than the binomial distribution; fewer PMFs are concentrated around the mean value.

The binomial distribution does not depend on the size of the sample as long as the expected value of p remains the same. As the size of the sample increases, of course with additional cost, the confidence in the value of p increases. This additional information cannot be used in the binomial distribution. To amplify the point, consider another case where $n = 235$ and $m = 190$, giving the same value of p considered previously; however, the sample size is only one-tenth in this case. The CDF for the hyperbinomial distribution in this case is given in Column 4 of Table 4.1. There would be no change in the CDF values for the binomial distribution, but Columns 3 and 4 according to the hyperbinomial distribution are different. This clearly indicates that the size of the sample is an important parameter in estimating the value of p, and the hyperbinomial distribution can effectively use this information. In decision making, engineers should consider the cost of obtaining additional samples and the benefit derived from that additional information.

The effect of sample size in estimating the value of p can be explained another way. As noted earlier, p has a beta distribution, and the parameters of the distribution depend on the sample size (Equations 4.29a and b). The PDF of p is plotted in Figure 4.5 when

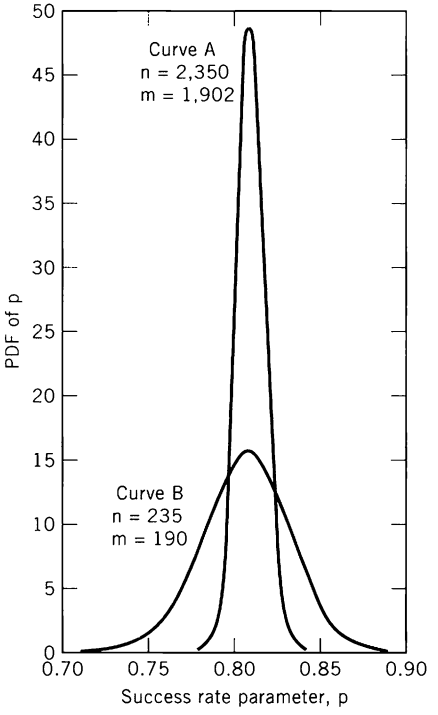


Figure 4.5 Distribution of Success Rate Parameter p

the sample size n is 2,350 (Curve A) and 235 (Curve B). In both cases, the mean value of p is 0.809. It is clear from the curves that when the sample size is bigger, the PDF is much more steep and more concentrated around the mean value. The bigger sample size would certainly increase the confidence in the prediction. When all the samples are tested, the beta distribution will become a spike at the mean value.

Considering the practical aspects of the problem presented here, some of the welds in a large project are expected to be bad. This is taken into account through the conservatism in the design specifications, so the structure is not expected to be unsafe. However, locating these defective welds is very difficult. There is no deterministic solution to the problem other than destroying the structure and rebuilding it. And even then, the new structure would undoubtedly contain some bad welds.

4.5 EXTREME VALUE DISTRIBUTIONS

4.5.1 Introduction

In many engineering applications, the extreme values of random variables are of special importance. The largest or smallest values of random variables may dictate a particular design. Wind speeds are recorded continuously at airports and weather stations. Obviously, the voluminous information collected cannot be used directly in engineering design. The maximum wind speeds per hour, day, month, year, or other period can be used for this purpose. Usually, the information on yearly maximum wind speed is used in the engineering profession. Thus, for every year of recorded data, the maximum wind speed is noted. If data are collected for several years, the design wind speed can be established statistically to ensure that it will not be exceeded within the design life of the structure with a specific probability level. If the design wind speed has a 50-year return period, then the probability that the wind speed will exceed the design value in a year is $1/50 = 0.02$. Design earthquake loads, flood levels, and so forth are also determined in this way. In all these cases, the peak or maximum value of a random variable during certain intervals is of interest. In some cases, the minimum value of a random variable is also of interest for design applications. For example, when a large number of identical devices are manufactured, such as calculators or cars, their minimum service lives are of great interest to consumers. Some of them could be subjected to accelerated testing to determine their life, and the probability distribution of life could be constructed. Then the minimum service life could be established so that it does not fall below an acceptable number with a predetermined probability level. Therefore, extreme value statistics have received a lot of attention for engineering design applications.

In constructing an extreme value distribution, an underlying random variable with a particular distribution is necessary. If different sets of samples are obtained (through physical or numerical experimentation), one can select the extreme values from each sample set—either the maximum or the minimum values—and then construct a different distribution for the extreme values. Therefore, the underlying distribution of a variable governs the form of the corresponding extreme value distribution.

The detailed mathematical aspects of extreme value distributions can be found elsewhere (Gumbel, 1958; Castillo, 1988). Only the essential concept, emphasizing engineering applications, is presented very briefly here.

4.5.2 Concept of Extreme Value Distributions

Let X be a random variable with some known distribution function. If there are n samples for the random variable X, then the extreme values of the sample, such as the minimum value Y_1, or the maximum value Y_n, may be of interest. Y_1 and Y_n can be defined as

$$Y_n = \max(X_1, X_2, \ldots, X_n) \tag{4.34}$$

$$Y_1 = \min(X_1, X_2, \ldots, X_n) \tag{4.35}$$

If different sets of samples of the same size n are obtained for X, each set will have different minimum and maximum values. Using all these sets, distribution functions for the minimum and maximum values can be constructed. The cumulative distribution function (CDF) of the largest value Y_n can derived as

$$F_{Y_n}(y) = P(Y_n \le y) = P(X_1 \le y, X_2 \le y, \ldots, X_n \le y). \tag{4.36}$$

For identically distributed and statistically independent X_i's, Equation 4.36 becomes

$$F_{Y_n}(y) = \left[F_X(y)\right]^n. \tag{4.37}$$

Similarly, the CDF of the smallest value Y_1 can be derived as

$$P(Y_1 > y) = P(X_1 > y, X_2 > y, \ldots, X_n > y) = 1 - F_{Y_1}(y). \tag{4.38}$$

Again, for identically distributed and statistically independent X_i's, Equation 4.38 becomes

$$F_{Y_1}(y) = 1 - \left[1 - F_X(y)\right]^n. \tag{4.39}$$

The basic idea of these derivations is that if the largest value Y_n is less than some quantity y, then all the sample values X_1, X_2, etc., should also be less than y; similarly, for the smallest value Y_1, if Y_1 is greater than some quantity y, then all of the samples X_1, X_2 up to X_n should be greater than y.

EXAMPLE 4.16

Suppose 10 cracks are detected in a beam in a bridge deck segment. For the purpose of illustration, assume the crack sizes are normally distributed with a mean value of 0.5 inch and a COV of 0.1. What is the probability that the maximum crack size is less than 0.6 inch?

This is an extreme (maximum) value problem. The distribution of the largest value and therefore the probability of the largest crack being less than 0.6 inch can be calculated as discussed below.

Let X be the random variable denoting the size of a crack. From the data, X is a normal random variable with $\mu_X = 0.5$ inch and $\delta_X = 0.1$. Using Equation 4.34, we can express the CDF of Y_{10} = largest value among 10 samples of X as

$$F_{Y_{10}}(y) = P(Y_{10} \le y) = \left[P(X \le y)\right]^{10}.$$

The probability that the size of any crack is less than 0.6 inch is

$$P(X \le 0.6) = \Phi\left(\frac{0.6 - 0.5}{0.05}\right) = \Phi(2.0) = 0.9772.$$

Therefore, the probability that the maximum crack size is less than 0.6 inch is

$$P(Y_{10} \le 0.6) = [P(X \le 0.6)]^{10} = 0.794.$$

4.5.3 Asymptotic Distributions

In Equations 4.36 to 4.39, as the sample size n grows larger and approaches infinity, the distribution of the largest or the smallest values may asymptotically approach a mathematical distribution function in some cases if the samples are identically distributed and statistically independent. Some of these asymptotic distributions have a wide range of applications in engineering problems. Gumbel (1958) classified three types of asymptotic extreme value distributions for both minima and maxima, labeling them as Type I, Type II, and Type III extreme value distributions. The Type I extreme value distribution of the largest value is also referred to as EVD (extreme value distribution) in mechanical reliability engineering applications. The distribution of maxima in sample sets from a population with a normal distribution will asymptotically converge to this distribution. This distribution is used to model environmental phenomena such as wind loads and flood levels. The Type II extreme value distribution of the largest value is also used to model extreme environmental phenomena such as earthquake loads and may result from sample sets from a lognormal distribution. The Type III extreme value distribution, which is referred to as the Weibull distribution in the case of the smallest value, may be obtained by the convergence of most of the commonly known distributions that have a lower bound. It is commonly used to describe material strengths and time to failure of electronic and mechanical devices and components. These distributions are discussed in the following sections.

Extreme value distributions are treated no differently than any other distributions discussed earlier. An extreme value distribution can be uniquely defined in terms of its PDF or CDF and the parameters of the distributions. In most cases, the parameters can be estimated from the information on the mean, variance, or coefficient of variation of the random variable. Once an extreme value distribution is uniquely defined, probabilistic information can be extracted from it using the procedure discussed in the previous sections. In the following sections, some of the commonly used extreme value distributions are discussed, emphasizing their basic definition.

4.5.4 The Type I Extreme Value Distribution

The CDF of the Type I asymptotic form of the distribution of the largest value, also referred to as the Gumbel distribution or simply the EVD, can be expressed as

$$F_{Y_n}(y_n) = \exp\left[-e^{-\alpha_n(y_n - u_n)}\right] \tag{4.40}$$

where u_n is the characteristic largest value of the initial variable X, and α_n is an inverse measure of dispersion of the largest value of X. The corresponding probability density function PDF can be shown to be

$$f_{Y_n}(y_n) = \alpha_n e^{-\alpha_n(y_n - u_n)} \exp\left[-e^{-\alpha_n(y_n - u_n)}\right], \quad -\infty < y_n < +\infty. \tag{4.41}$$

The parameters u_n and α_n are related to the mean and standard deviation of the extreme value variable Y_n as

$$\alpha_n = \frac{1}{\sqrt{6}}\left(\frac{\pi}{\sigma_{Y_n}}\right), \text{ and } u_n = \mu_{Y_n} - \frac{0.5772}{\alpha_n}. \tag{4.42}$$

As mentioned earlier, the Type I extreme value distribution for the largest value is commonly used for modeling environmental loads such as winds and floods. Another use of the Type I extreme value distribution for maxima is in aircraft design, where the peak gust velocity experienced by an aircraft during every 1,000 hours of operation is considered to be a Type I largest variable. Another common example is the maximum water level in a year at a specified location in a river. This is an important variable in the design of flood control, water supply, and irrigation systems.

For the smallest value of an initial variable X, the corresponding Type I asymptotic form for the CDF is

$$F_{Y_1}(y_1) = 1 - \exp\left[-e^{\alpha_1(y_1 - u_1)}\right]. \tag{4.43a}$$

The corresponding PDF is

$$f_{Y_1}(y_1) = \alpha_1 e^{\alpha_1(y_1 - u_1)} \exp\left[-e^{\alpha_1(y_1 - u_1)}\right], \quad -\infty < y_1 < +\infty. \tag{4.43b}$$

In these equations, the parameters are defined as u_1, which is the characteristic smallest value of the initial variable X, and α_1, which is an inverse measure of dispersion of the smallest value of X. u_1 and α_1 are related to the mean and standard deviation of Y_1 as

$$\alpha_1 = \frac{1}{\sqrt{6}}\left(\frac{\pi}{\sigma_{Y_1}}\right), \text{ and } u_1 = \mu_{Y_1} + \frac{0.5772}{\alpha_1}. \tag{4.44}$$

In general, the Type I asymptotic form is obtained by the convergence of distributions with an exponential tail. For example, the PDF of the Gaussian distribution has an exponential decaying term and therefore an exponential tail in the extreme directions. The extreme values of a variable with a Gaussian distribution will have a Type I distribution.

EXAMPLE 4.17

The data on maximum annual wind velocity V_n at a site have been compiled for n years, and its mean and standard deviation are estimated to be 61.3 mph and 7.52 mph, respectively. Assuming that V_n has a Type I extreme value distribution, what is the probability that the maximum wind velocity will exceed 100 mph in any given year?

SOLUTION

Equation 4.42 can be used to calculate the two parameters u_n and α_n of the Type I extreme value distribution as

$$\alpha_n = \frac{1}{\sqrt{6}}\left(\frac{\pi}{\sigma_{Y_n}}\right) = 0.17055, \text{ and } u_n = \mu_{Y_n} - \frac{0.5772}{\alpha_n} = 57.9157.$$

Therefore, from Equation 4.40, the probability that the maximum wind velocity is greater than 100 mph is

$$P(Y_n > 100) = 1 - F_{Y_n}(100) = 1 - \exp\left[-e^{-0.17055(100-57.9157)}\right] = 0.000763.$$

This type of analysis may also be used to calculate design values for various types of engineering applications. Suppose the design wind speed with a return period of 100 years needs to be estimated for a particular site. With V_d denoted as the design wind speed to be estimated, the probability that it will be exceeded in a given year is 1/100 = 0.01. Thus,

$$P(Y_n > V_d) = 1 - F_{Y_n}(V_d) = 0.01$$

or

$$1 - \exp\left[-e^{-0.17055(V_d - 57.9157)}\right] = 0.01$$

or $V_d = 84.89$ mph.

4.5.5 The Type II Extreme Value Distribution

The CDF of the Type II asymptotic form for the largest value, also referred to as the Fréchet distribution, can be shown to be

$$F_{Y_n}(y_n) = \exp\left[-\left(\frac{v_n}{y_n}\right)^k\right]. \tag{4.45}$$

The corresponding PDF is

$$f_{Y_n}(y_n) = \frac{k}{v_n}\left(\frac{v_n}{y_n}\right)^{k+1} \exp\left[-\left(\frac{v_n}{y_n}\right)^k\right], \quad y_n \geq 0, \quad k > 2 \tag{4.46}$$

where v_n and k are the parameters of the distribution; v_n is the characteristic largest value of the underlying variable X; and k, the shape parameter, is a measure of dispersion.

The Type II asymptotic form is obtained as n goes to infinity from an initial distribution that has a polynomial tail in the direction of the extreme value. Note the difference between this and Type I, which converges from an exponential tail. The Type II distribution requires a polynomial tail, and therefore a lognormal distribution converges to a Type II asymptotic form for the largest value.

It is also interesting to note the relationship between the Type I and Type II forms. It was observed in Sections 4.2.1 and 4.2.2 that the normal and lognormal distributions are related to each other; that is, if a variable has lognormal distribution, the natural log of that variable has a normal distribution. The Type I and Type II extreme value distributions may be obtained through the asymptotic convergence of these two initial distributions. Therefore, if Y_n has a Type II asymptotic distribution with parameters v_n and k, then $\ln Y_n$ will have a Type I asymptotic form with parameters $u_n = \ln v_n$ and $\alpha_n = k$.

For the Type II distribution of maxima, the mean, standard deviation, and COV of Y_n are related to the distribution parameters v_n and k as follows:

$$\mu_{Y_n} = v_n \Gamma\left(1 - \frac{1}{k}\right), \quad k > 1 \tag{4.47a}$$

$$\sigma_{Y_n}^2 = v_n^2\left[\Gamma\left(1-\frac{2}{k}\right)-\Gamma^2\left(1-\frac{1}{k}\right)\right], \quad k>2 \tag{4.47b}$$

and

$$1+\delta_{Y_n}^2 = \frac{\Gamma\left(1-\frac{2}{k}\right)}{\Gamma^2\left(1-\frac{1}{k}\right)}, \quad k>2. \tag{4.47c}$$

In these equations, Γ is the gamma function and can be estimated by Appendix 2. A plot of $(1+\delta_{Y_n}^2)$ versus $1/k$ is given in Figure 4.6. This figure can be helpful in solving problems, as will be shown later with examples.

The CDF and PDF of the Type II asymptotic form of the smallest value can be shown to be

$$F_{Y_1}(y_1) = 1-\exp\left[-\left(\frac{v_1}{y_1}\right)^k\right] \tag{4.48}$$

and

$$f_{Y_1}(y_1) = \frac{k}{v_1}\left(\frac{v_1}{y_1}\right)^{k+1}\exp\left[-\left(\frac{v_1}{y_1}\right)^k\right], \quad y_1 \le 0, \quad v_1 > 0 \tag{4.49}$$

where the parameter v_1 is the characteristic smallest value of the initial variable X and k is the shape parameter, an inverse measure of dispersion. The comment regarding the relationship between Type I and Type II asymptotic forms of the largest value also

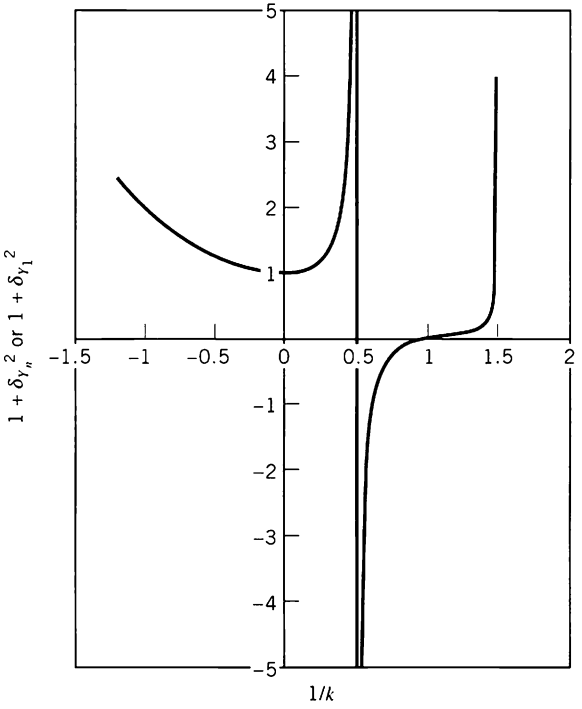


Figure 4.6 Relationship Between COV of Y_n (or Y_1) and $1/k$

holds for the asymptotic forms of the smallest value. Therefore, if Y_1 has a Type II asymptotic distribution with positive parameters v_1 and k, then ln Y_1 has a Type I asymptotic distribution with parameters $u_1 = \ln v_1$ and $\alpha_1 = k$.

For the Type II distribution of minima, the mean, standard deviation, and COV of Y_1 are related to the distribution parameters v_1 and k as follows:

$$\mu_{Y_1} = v_1 \Gamma\left(1 - \frac{1}{k}\right), \quad k > 1 \tag{4.50a}$$

$$\sigma_{Y_1}^2 = v_1^2 \left[\Gamma\left(1 - \frac{2}{k}\right) - \Gamma^2\left(1 - \frac{1}{k}\right)\right], \quad k > 2 \tag{4.50b}$$

and

$$1 + \delta_{Y_1}^2 = \frac{\Gamma\left(1 - \frac{2}{k}\right)}{\Gamma^2\left(1 - \frac{1}{k}\right)}, \quad k > 2. \tag{4.50c}$$

By comparing Equations 4.47c and 4.50c, we can observe that the relationship between the COV of Y_n and k is identical to the relationship between the COV of Y_1 and k. Thus, Figure 4.6 is also applicable for the Type II distribution of minima.

EXAMPLE 4.18

Suppose the example on the annual maximum wind velocity considered in Example 4.17 is to be modeled using a Type II extreme value distribution of maxima. What is the probability that the maximum wind velocity will exceed 100 mph in any given year?

SOLUTION

The parameters of the Type II distribution have to be determined first, using Equations 4.47a and 4.47b. From the previous example, the mean and the standard deviation of maximum wind velocity are 61.3 mph and 7.52 mph, respectively. Therefore, $\delta_{Y_n} = 7.52/61.3 = 0.123$. From Equation 4.47c,

$$1 + (0.123)^2 = 1.015 = \frac{\Gamma\left(1 - \frac{2}{k}\right)}{\Gamma^2\left(1 - \frac{1}{k}\right)}.$$

Referring to Figure 4.6, and considering k is greater than 2, we can estimate k to be 10. From Equation 4.47a,

$$61.3 = v_n \Gamma\left(1 - \frac{1}{k}\right) = v_n \Gamma(0.9).$$

Using Appendix 2, we find $\Gamma(0.9)$ to be 1.0686. Thus,

$$v_n = \frac{61.3}{1.0686} = 57.36.$$

Therefore, using Equation 4.45, we find the probability that the maximum wind velocity will exceed 100 mph in any given year to be

$$P(Y_n > 100) = 1 - \exp\left[-\left(\frac{57.36}{100}\right)^{10}\right] = 1 - 0.99615 = 0.00385.$$

4.5.6 The Type III Extreme Value Distribution

Both the Type I and Type II asymptotic distributions are limiting forms of the distribution of extreme values from initial distributions that are unbounded in the direction of the extreme value. In contrast, the Type III asymptotic form represents a limiting distribution of the extreme values from initial distributions that have a finite upper or lower bound value.

For the largest value, the Type III asymptotic CDF can be written as

$$F_{Y_n}(y_n) = \exp\left[-\left(\frac{\omega - y_n}{\omega - w_n}\right)^k\right]. \tag{4.51}$$

The corresponding PDF is

$$f_{Y_n}(y_n) = \frac{k}{\omega - w_n}\left(\frac{\omega - y_n}{\omega - w_n}\right)^{k-1} \exp\left[-\left(\frac{\omega - y_n}{\omega - w_n}\right)^k\right], \quad y_n \le \omega \tag{4.52}$$

where ω is the upper bound of the initial distribution, that is, $F_X(\omega) = 1.0$, and w_n and k are the parameters of the distribution. w_n is the characteristic largest value of X, and is defined by

$$F_X(w_n) = 1 - \frac{1}{n}, \tag{4.53}$$

and k is a shape parameter.

The mean and variance of Y_n are related to the parameters w_n and k as follows:

$$\mu_{Y_n} = \omega - (\omega - w_n)\Gamma\left(1 + \frac{1}{k}\right) \tag{4.54a}$$

and

$$\sigma^2_{Y_n} = \mathrm{Var}(\omega - Y_n) = (\omega - w_n)^2\left[\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)\right]. \tag{4.54b}$$

Equations 4.54a and 4.54b can be used to show that

$$1 + \left(\frac{\sigma_{Y_n}}{\omega - \mu_{Y_n}}\right)^2 = \frac{\Gamma\left(1 + \frac{2}{k}\right)}{\Gamma^2\left(1 + \frac{1}{k}\right)}. \tag{4.54c}$$

The relationship is shown graphically in Figure 4.7.

The CDF of the Type III asymptotic distribution of the smallest value from an initial distribution with a lower limit ε, that is, $F_X(\varepsilon) = 0$, is

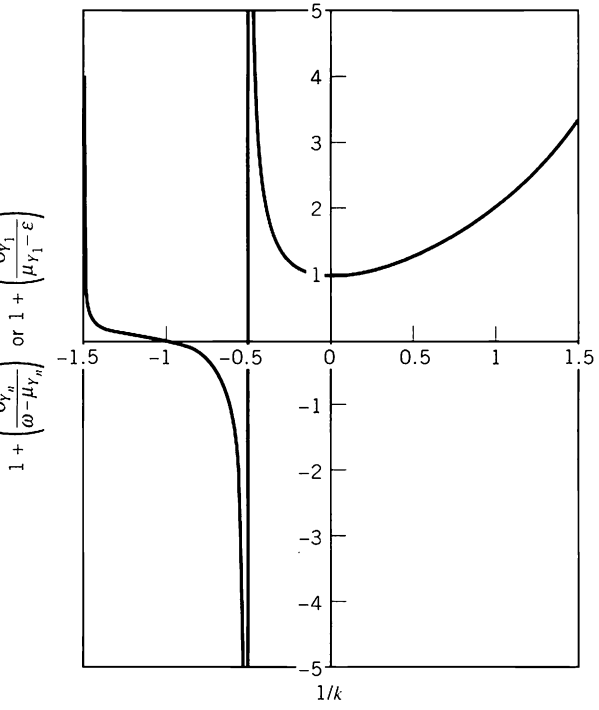


Figure 4.7 Relationship Between the Mean and Variance of Y_n (or Y_1) and $1/k$

$$F_{Y_1}(y_1) = 1 - \exp\left[-\left(\frac{y_1 - \varepsilon}{w_1 - \varepsilon}\right)^k\right]. \tag{4.55}$$

The corresponding PDF is

$$f_{Y_1}(y_1) = \frac{k}{w_1 - \varepsilon}\left(\frac{y_1 - \varepsilon}{w_1 - \varepsilon}\right)^{k-1} \exp\left[-\left(\frac{y_1 - \varepsilon}{w_1 - \varepsilon}\right)^k\right], \quad y_1 \geq \varepsilon \tag{4.56}$$

where the w_1 and k are the parameters of the distribution. w_1 is the characteristic smallest value defined as

$$F_X(w_1) = \frac{1}{n} \tag{4.57}$$

and k is the shape parameter.

The mean and variance of Y_1 are related to the parameters w_1 and k as follows:

$$\mu_{Y_1} = \varepsilon + (w_1 - \varepsilon)\Gamma\left(1 + \frac{1}{k}\right) \tag{4.58a}$$

and

$$\sigma^2_{Y_1} = \mathrm{Var}(Y_1 - \varepsilon) = (w_1 - \varepsilon)^2\left[\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)\right]. \tag{4.58b}$$

Equations 4.58a and 4.58b can be used to show that

$$1+\left(\frac{\sigma_{Y_1}}{\mu_{Y_1}-\varepsilon}\right)^2=\frac{\Gamma\left(1+\frac{2}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)}. \tag{4.58c}$$

Comparing Equations 4.54c and 4.58c shows that Figure 4.7 is also appropriate to represent Equation 4.58c.

EXAMPLE 4.19

A number of L-shaped structural steel sections are rolled in a steel mill to be used as members in bridge trusses. The distribution of their minimum axial load capacity is of interest. Suppose the minimum load capacity can be modeled with a Type III distribution of the smallest value, with a mean value of 300 kips and a COV of 0.15. Also, assume that the load capacity has a lower bound of 100 kips. Determine the probability that the minimum load capacity will be less than 200 kips.

SOLUTION

Using $\varepsilon = 100$, we can calculate the left-hand side of Equation 4.58c as

$$1+\left(\frac{\sigma_{Y_1}}{\mu_{Y_1}-\varepsilon}\right)^2=1+\left(\frac{0.15\times 300}{300-100}\right)^2=1.0506.$$

Using Figure 4.7, $1/k = 0.20$, or $k = 5.0$. Using Equation 4.58a,

$$300=100+(w_1-100)\Gamma\left(1+\frac{1}{5.0}\right)$$

or

$$200=(w_1-100)\Gamma(1.20)$$

or

$$w_1=100+\frac{200}{\Gamma(1.20)}=100+\frac{200}{0.9182}=317.82.$$

Note $\Gamma(1.20) = 0.9182$ is calculated using Appendix 2. Therefore, Equation 4.55 can be used to calculate the probability that the minimum load capacity is less than 200 kips:

$$P(Y_1\le 200)=F_{Y_1}(200)=1-\exp\left[-\left(\frac{200-100}{317.82-100}\right)^{5.0}\right]=0.0202.$$

The Type III asymptotic distribution of the smallest value, developed by Weibull in connection with the study of fatigue and fracture of materials, is known as the Weibull distribution. The preceding equation is a three-parameter representation of the Weibull distribution. A two-parameter form of this distribution is commonly used in mechani-

cal and electronic component life estimation. It is obtained by setting ε to be zero in Equation 4.55, reflecting the physical nature of the problem.

The CDF of the two-parameter Weibull distribution is

$$F_{Y_1}(y_1) = 1 - \exp\left[-\left(\frac{y_1}{w_1}\right)^k\right]. \tag{4.59}$$

The corresponding PDF is

$$f_{Y_1}(y_1) = \frac{k}{w_1}\left(\frac{y_1}{w_1}\right)^{k-1} \exp\left[-\left(\frac{y_1}{w_1}\right)^k\right], \quad y_1 \geq 0. \tag{4.60}$$

In these two equations, k and w_1 have to be positive values.

For the two-parameter Weibull distribution, the mean value and the coefficient of variation are related to the parameters k and w_1 as follows:

$$\mu_{Y_1} = w_1 \Gamma\left(1 + \frac{1}{k}\right) \tag{4.61a}$$

and

$$\delta_{Y_1} = \left[\frac{\Gamma\left(1 + \frac{2}{k}\right)}{\Gamma^2\left(1 + \frac{1}{k}\right)} - 1\right]^{\frac{1}{2}} \tag{4.61b}$$

where $\Gamma(\ )$ is the gamma function. If the mean and coefficient of variation are known, the following approximation can be used to compute the parameters k and w_1 for practical applications:

$$k = \delta_{Y_1}^{-1.08} \tag{4.62a}$$

and

$$w_1 = \frac{\mu_{Y_1}}{\Gamma\left(1 + \frac{1}{k}\right)}. \tag{4.62b}$$

4.5.7 Special Cases of Two-Parameter Weibull Distribution

Two special cases of the two-parameter Weibull distribution used widely in engineering are the exponential distribution and the Rayleigh distribution. In Equation 4.59, if the random variable Y_1 is denoted by another random variable X, and $k = 1$, and $1/w_1 = \nu$, it results in an exponential distribution. The CDF and the PDF of an exponential distribution are given by Equations 4.26 and 4.27, respectively. Note that ν is the parameter of the exponential distribution and is the reciprocal of the mean or the standard deviation of the random variable X, as shown in Table 5.6 in Chapter 5. The exponential distribution is commonly used in the reliability analysis of electronic and mechanical devices. In Equations 4.26 and 4.27, if time to failure T of a unit has an exponential distribution, then ν is the failure occurrence rate.

Similarly, a two-parameter Weibull distribution with $k = 2$ and $w_1 = \sqrt{2}\alpha$ results in a Rayleigh distribution. For example, wave heights are modeled with a Rayleigh distribution. The PDF of the Rayleigh distribution is given in Table 5.6 in Chapter 5. Its parameter w_1 can be estimated from the information on the mean and variance of X, as shown in Table 5.6. An example of the use of this distribution is in the description of the peaks in a narrow-band stationary Gaussian random process.

4.6 OTHER USEFUL DISTRIBUTIONS

In the previous sections, some of the commonly used continuous and discrete random variables were identified and the calculation of probability for each was described. Some other distributions that will be used in later chapters are Student's t-distribution and chi-square distribution, which are briefly introduced in Chapter 5. Several other standard distributions are available for engineering applications. The exact forms of their PDFs can be determined by referring to the literature. Once their PDFs have been defined uniquely, their probability estimation procedures are similar to the distributions discussed here.

The selection of one distribution over the others and the selection of parameters to describe a distribution uniquely will be discussed in Chapter 5. However, in many engineering problems, there is not enough information available to justify the use of a particular standard distribution. Based on limited experience, an engineer may have some idea of the lower and upper limits of a random variable, but there may not be enough data available between these two limits to justify a specific distribution. In this situation, any distribution can be used, such as a uniform distribution or one of the many different forms of triangular or trapezoidal distributions shown in Table 4.2. Of course, all the criteria to qualify them as legitimate PDFs must be satisfied, as discussed in Section 3.3.1. The parameters for these nonstandard distributions cannot be calculated in terms of mean and standard deviation based on sample information, because it is not available. However, they can be calculated from the assumed shape of the distribution. As discussed in Section 3.3.3, the mean of a random variable represents the centroidal distance, and the variance is the first moment of inertia of the area about the centroidal axis. For an assumed shape, these values can be easily calculated and are shown in Table 4.2.

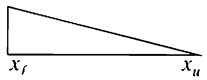
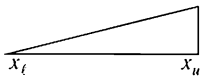
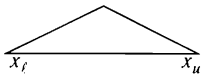
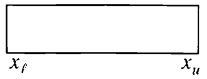
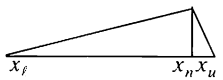
4.7 CONCLUDING REMARKS

Some commonly used distributions, both continuous and discrete, and procedures to calculate the probability of events using them are discussed in this chapter. The use of these distributions to solve some practical problems is presented to show their implementation potential.

In many engineering applications, the largest or smallest values of random variables may dictate a particular design. To model them, some of the commonly used extreme value distributions are discussed.

The information presented here is expected to provide sufficient background to calculate probabilities of events using commonly used distributions. Available computer programs can also be used for this purpose.

Table 4.2 Mean and Coefficient of Variation of X Corresponding to Different Distributions Assumed Over Its Range

Case no.	Distribution	Mean of X	COV of X
1		$\frac{1}{3}(2x_\ell + x_u)$	$\frac{1}{\sqrt{2}} \frac{x_u - x_\ell}{2x_\ell + x_u}$
2		$\frac{1}{3}(x_\ell + 2x_u)$	$\frac{1}{\sqrt{2}} \frac{x_u - x_\ell}{x_\ell + 2x_u}$
3		$\frac{1}{2}(x_\ell + x_u)$	$\frac{1}{\sqrt{6}} \frac{x_u - x_\ell}{x_u + x_\ell}$
4		$\frac{1}{2}(x_\ell + x_u)$	$\frac{2}{\sqrt{12}} \frac{x_u - x_\ell}{x_u + x_\ell}$
5		$\frac{1}{3}(x_\ell + x_u + x_n)$	$\frac{1}{\sqrt{2}} \frac{\sqrt{x_\ell^2 + x_u^2 + x_n^2 - x_\ell x_u - x_\ell x_n - x_u x_n}}{(x_\ell + x_u + x_n)}$

4.8 PROBLEMS

4.1 The breaking strength, R, of a cable can be assumed to be a normal random variable with a mean value of 80 kip and a standard deviation of 20 kip.

(a) If a load, P, of magnitude 60 kip is hung from the cable, calculate the probability of failure of the cable.

(b) The magnitude of P cannot be determined with certainty. Suppose it could be either 40 kip or 60 kip, and the corresponding PMFs are shown in Figure P4.1. Calculate the probability of failure of the cable.

(c) If the cable breaks, what is the probability that the load was 40 kip?

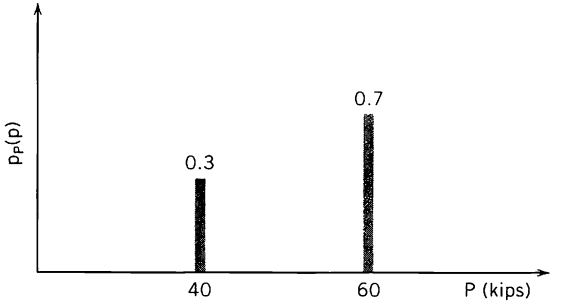


Figure P4.1 PMF of P

4.2 The magnitude of a load acting on a structure can be modeled by a normal distribution with a mean of 100 kip and a standard deviation of 20 kip.

(a) If the design load is considered to be the 90th percentile value, determine the design load.

(b) If the design load is considered to be the mean + 2 standard deviation value, what is the probability that it will be exceeded?

(c) A load of magnitude less than zero is physically illogical; calculate its probability. Is a normal distribution appropriate to model the load?

4.3 The capacity of an isolated spread footing foundation under a column is modeled by a normal distribution with a mean of 300 kip and a COV of 20%. Suppose the column is subjected to a dead load of 100 kip and a live load of 150 kip.

(a) Calculate the probability of failure of the foundation under dead load only.

(b) Calculate the probability of failure of the foundation under the combined action of dead and live loads.

(c) If the probability of failure of the foundation needs to be limited to 0.001, and the dead load of 100 kip cannot be changed, what is the maximum amount of live load that can be applied to the foundation?

4.4 The annual rainfall for a city is assumed to be normally distributed with a mean of 100 cm, and its mean ±3 standard deviation values are estimated to be 160 and 40 cm, respectively.

(a) Calculate the standard deviation of the annual rainfall.

(b) What is the probability that the rainfall will be less than 0?

(c) What is the probability that the annual rainfall will be within the ±3 standard deviation values?

(d) Is normal distribution appropriate in this case?

4.5 Solve Parts (a) and (b) of Problem 4.1, assuming that the breaking strength of the cable is a lognormal variable with the same mean and standard deviation.

4.6 Solve all three parts of Problem 4.2, assuming the load to be a lognormal random variable with the same mean and standard deviation.

4.7 Solve all three parts of Problem 4.3, assuming the capacity of the foundation is modeled by a lognormal random variable with the same mean and COV.

4.8 The compressive strength of concrete delivered by a supplier can be modeled by a lognormal random variable. Its mean and the coefficient of variation are estimated to be 4.7 ksi and 0.21, respectively.

(a) If the 10th percentile value is the design value, calculate the value of the compressive strength to be used in a design.

(b) Suppose the COV of the compressive strength is reduced to 0.10 without affecting its mean value by introducing quality control procedures. Calculate the design value of the compressive strength if it is assumed to be the 10th percentile value.

(c) By comparing the results obtained in Parts (a) and (b), discuss whether quality control measures are preferable.

4.9 The northbound train traffic in a subway station between 7 and 8 a.m. on a typical workday is studied. Trains are supposed to arrive every 5 minutes. Collected data indicate that trains generally arrive at the station with an average delay of 1 minute and a variance of 2.0 min^2. Assume that the delay of each train is statistically independent and lognormally distributed, and if a train arrives within 30 seconds of the scheduled time it is not considered to be late.

(a) What is the probability that a train will arrive late at this station?

(b) What is the probability that the first train to arrive on time will be the third train?

(c) What is the probability that no train will arrive at the station on time during the 1 hour of peak traffic?

(d) If a train is late, what is the probability that it will arrive within 1 minute of the scheduled time?

4.10 A contractor purchases a large number of bolts in one batch for future use. Based on past experience, the contractor estimates that the tensile strength of each bolt can be modeled by a lognormal variable with a mean of 80 kip and a COV of 0.15. A bolt must carry at least 60 kip to be acceptable. It is impractical to test all the bolts for strength. For quality control purposes, the contractor proposes the following three inspection schemes: (a) all 5 bolts selected at random from the batch must pass the test, (b) at least 6 of the first 7 bolts tested must pass the test, and (c) at least 8 of the first 10 bolts tested must pass the test. Which inspection scheme is the most severe from the supplier's point of view?

4.11 The relative density of a homogeneous soil deposit is measured and found to have a mean value of 0.80 and a COV of 0.20. From a theoretical point of view, the relative density can be between 0 and 1. Suppose the relative density has a beta distribution.

(a) Define the PDF of the relative density.

(b) What is the probability that the relative density of the soil deposit is greater than 0.90?

4.12 The travel time T between home and office is expected to be between 20 and 40 minutes depending upon traffic. Based on experience, the average travel time is 30 minutes and the corresponding variance is 20 min^2.

(a) Determine the PDF of T. Hint: Assume it is a beta distribution.

(b) What is the probability that T will exceed 30 minutes on a particular day?

4.13 The probability that the maximum temperature in a typical summer day will exceed 100° F in a desert city is estimated to be 0.30.

(a) What is the probability that the maximum temperature will not exceed 100° F in the next 7 days?

(b) What is the probability that the maximum temperature will exceed 100° F in only 2 of the next 7 days?

(c) What is the probability that the maximum temperature will exceed 100° F at least 3 days in the next 7 days?

(d) What is the probability that the maximum temperature will exceed 100° F in at most 2 days in the next 7 days?

4.14 The probability of damage to a structure due to fire, p, is estimated to be 0.05 per year. Assume the design life of the structure is 50 years.

(a) What is the probability that the structure will not be damaged by fire during its design life?

(b) What is the probability that the structure will be damaged due to fire in the 10th year?

(c) If the insurance company requires that the maximum risk of damage to the structure be limited to 0.10 during its lifetime, calculate the maximum permissible value of p.

4.15 The mean compressive strength of a batch of concrete is found to have a lognormal distribution with a mean value of 5,000 psi and a standard deviation of 500 psi. The minimum required strength is 4,000 psi. Five cylinders from this batch are tested.

(a) What is the probability that at least one cylinder will fail?

(b) What is the probability that two cylinders will fail?

(c) What is the probability that the fifth cylinder will fail and the others will pass?

4.16 The annual precipitation in inches per year in Tucson, Arizona, for the past thirty years are as follows: 11.60, 7.19, 12.69, 11.86, 14.81, 8.07, 11.15, 8.00, 9.55, 11.02, 19.54, 8.63, 12.33, 8.53, 16.55, 19.74, 18.40, 11.37, 10.55, 8.68, 9.62, 6.93, 14.80, 10.64, 14.76, 15.19, 14.56, 9.68, 11.13, and 4.35. Assume the annual precipitation follows a Poisson process.

(a) On average, how often will the annual precipitation exceed 12 in./yr?

(b) What is the probability that in the next 5 years, the annual precipitation will exceed 12 in./yr exactly twice?

(c) What is the probability that at least once in the next 5 years, the precipitation will exceed 12 in./yr?

(d) How will the probability in Part (b) change if the annual precipitation increases to 15 in.?

4.17 The PMF of the number of fires in a subdivision in a year is shown in Figure P4.17. More than two fires a year did not occur in the subdivision. During a fire, the mean property damage is expected to be $50,000 with a COV of 0.2. Assume that property damage can be modeled by a lognormal random variable, and the property damage for different fires is statistically independent.

(a) What is the probability of property damage exceeding $100,000 in a fire?

(b) What is the probability that none of the fires in a year will cause property damage exceeding $100,000?

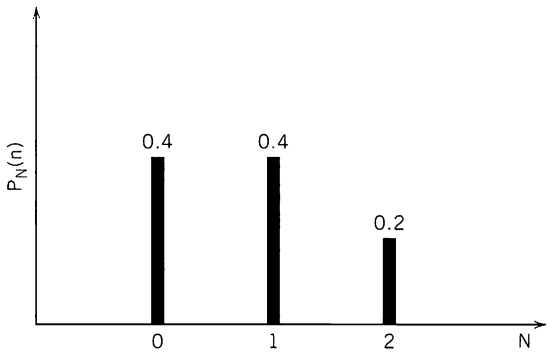


Figure P4.17 PMF of Number of Fires

4.18 On average, one damaging earthquake occurs in a county every 10 years. Assume the occurrence of earthquakes is a Poisson process in time.

(a) What is the probability of having at most two earthquakes in 1 year?

(b) What is the probability of having at least one earthquake in 5 years?

4.19 The available record indicates that there were two fires in the past 10 years in a subdivision.

(a) What is the probability that there will be no fire in the subdivision in the next 10 years?

(b) What is the probability that there will be at least one fire in the subdivision in the next 10 years?

(c) What is the probability that there will be at least two fires in the subdivision in the next 10 years?

4.20 The occurrence of floods in a county follows a Poisson process at an average rate of once in 20 years. The damage in each flood is lognormally distributed with a mean of $2 million and a COV of 25%. Assume that damage in any one flood is statistically independent of the damage in any other flood.

(a) What is the probability of more than two floods occurring in the county during the next 10 years?

(b) What is the probability that damage in the next flood will exceed \$3 million?

(c) What is the probability that damage in each of the next two floods will exceed \$3 million?

(d) What is the probability that none of the floods will cause damage exceeding \$3 million in the next 10 years?

4.21 The average car accident rate at a particular intersection is one per month. During an accident, the probability of personal injury is 0.30. Assume that accidents occur according to a Poisson process.

(a) What is the probability that there will be no accident in the intersection next year?

(b) What is the probability that there will be no personal injury in the next n accidents?

(c) What is the probability that there will be no personal injury in the intersection next year?

4.22 A bank employee notices that there are no customers waiting for service, and she decides to take a 5-minute break. On an average, one customer arrives every 2 minutes, in a Poisson process. It takes 1 minute to serve each customer.

(a) When she returns from the break, what is the probability that five customers are waiting?

(b) Suppose she returns and finds five customers waiting. What is the probability that she will be able to take another break in exactly 10 minutes?

4.23 A driver arrives at a T-junction, as shown in Figure P4.23, and notices 10 cars ahead of him at the stop sign. It takes 3 seconds for each car to clear the intersection. Cars in the other direction arrive at an average of one car every 10 seconds in a Poisson process. Cars in the two directions alternate, one at a time, in crossing the intersection. What is the probability that the driver will clear the intersection in 45 seconds or less?

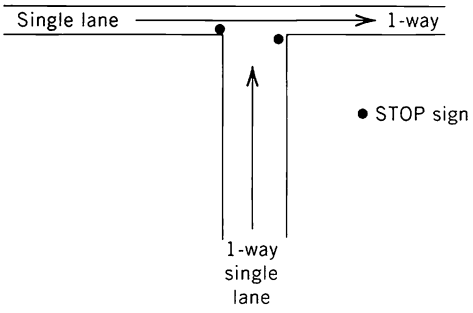


Figure P4.23 A T-junction

4.24 Suppose that on an average two tornadoes occur in 10 years in a county in Oklahoma. Further assume that the tornado-generated wind speed can be modeled by a lognormal random variable with a mean of 120 mph and a standard deviation of 12 mph.

(a) What is the probability that there will be at least one tornado next year?

(b) If a structure in the county is designed for wind speed of 150 mph, what is the probability that the structure will be damaged during such a tornado?

(c) What is the probability that the structure will be damaged by tornado next year? (Hint: Consider that any number of tornadoes can occur next year.)

4.25 Suppose the life of a light bulb can be modeled by an exponential distribution with an average life of 12 months. Suppose a maintenance worker checks the bulb every 6 months.

(a) What is the probability that the light bulb will need to be replaced at the first scheduled inspection?

(b) If the bulb is in good condition during the first scheduled inspection, what is the probability that it will be in good condition during the next scheduled inspection?

(c) If there are 10 light bulbs in a room, what is the probability that at least one of them needs replacement at the first scheduled inspection?

4.26 Structures in a county need to be designed for earthquake loading. After a detailed seismic risk analysis of the county, it is observed that the peak ground acceleration A can be modeled by an exponential distribution with a mean of 0.2g. If A exceeds 0.4g, structures in the county will suffer significant damage. Assume that earthquakes with A exceeding 0.4g occure once every 15 years in the county, and the damage from different earthquakes is statistically independent.

(a) What is the probability that there will be exactly two earthquakes where A exceeds 0.4g in the next 50 years?

(b) What is the probability of significant structural damage in an earthquake?

(c) What is the probability of no significant structural damage due to earthquakes in a year?

(d) What is the probability of no significant structural damage due to earthquakes in the next 50 years?

4.27 Studs are generally used to develop composite action between the steel beams and concrete slab. In a particular building, 240 studs are tested, and 170 of them are found to be good. Suppose a typical beam contains 10 studs. Calculate the number of good studs in a beam with a 90% confidence level using the hyperbinomial distribution.

4.28 In Problem 4.27, suppose only 24 studs are tested, and 17 of them are found to be good. How many good studs are expected in a beam containing a total of 10 studs, at a confidence level of 90%?

4.29 The annual maximum stage height in a river channel is modeled using a Type I extreme value distribution of the largest value, with a mean value of 30 ft and a COV of 10%. The stage height at which flooding will occur is 40 ft. What is the probability that the annual maximum stage height will exceed this level?

4.30 A steel cable consists of eight high-strength steel strands. The strength of each strand can be modeled by a lognormal random variable with a mean of 50 kip and a COV of 10%. What is the probability that the weakest strand will have a strength less than 40 kip?

4.31 In a seismic hazard analysis, the magnitude of the earthquake (Richter's scale) is modeled with a Type II extreme value distribution of the largest value. In a certain geographical region, data have been collected for 50 years, and the annual maximum values of the earthquake magnitude have a mean value of 4.0 and a standard deviation of 2.0. It is estimated that an earthquake with a magnitude of 9.0 or more will devastate the region. What is the probability that the annual maximum magnitude will be greater than or equal to 9.0?

4.32 The safety of a statistically determinate truss structure is governed by the weakest member in the truss. (This is referred to as a weakest link system, or a series system. Refer to Chapter 8 for details.) The safety margin, defined as the ratio of the resistance to the applied load, of the weakest member in a particular truss is judged to have a mean value of 1.5 and a COV of 10% and is assumed to follow a Type II extreme value distribution of the smallest value. The member, and therefore the whole truss, will fail if the safety margin of the weakest member drops below 1.0. What is the probability of failure of the truss?

4.33 The minimum life of an automobile brake pad is modeled with a Type III extreme value distribution of the smallest value. From the available data, the parameters of distribution

are estimated to be $k = 2.5$, $w_1 = 24$ months, and $\varepsilon = 0$. What are the mean and standard deviation of the brake pad's life? What is the probability that the brake pad will last longer than 36 months?

4.34 The minimum fatigue life of rivets in a compressor airseal inlet of a gas turbine engine is modeled with a two-parameter Weibull distribution. During accelerated testing for the purpose of certification and approval, the mean value of minimum life is found to be 90 minutes, and the COV is 15%. The rivet design is unacceptable if the minimum life during accelerated testing is less than 50 minutes. What is the probability of nonacceptance of the rivet design?

4.35 For a very unusual project, an engineer estimates that construction time may vary between 30 and 50 days. However, the engineer has no prior knowledge and believes that the completion time is equally likely between 30 and 50 days. Use Table 4.2 to calculate the following:

(a) The mean completion time.

(b) The standard deviation, variance, and coefficient of variation of the completion time.

(c) The probability that the completion time will be greater than 40 days.

4.36 In Problem 4.35, if the completion time has a triangular distribution with a modal value of 45 days, calculate the following:

(a) The mean completion time.

(b) The standard deviation, variance, and coefficient of variance of the completion time.

(c) The probability that the completion time will be greater than 40 days.

4.37 A cofferdam needs to be built to facilitate construction of a bridge pier. The maximum monthly flood level X at the site is considered to have a triangular distribution, as shown in Case 5 in Table 4.2, with x_l, x_n, and x_u values of 2, 8, and 10 ft, respectively. The pier will take 8 months to construct.

(a) What would be the height of the cofferdam with reliability 0.90, that is, the cofferdam will not be flooded during the construction period with a probability of 0.9?

(b) What is the height of the cofferdam corresponding to the flood return period of 8 months?

Chapter 5

Determination of Distributions and Parameters from Observed Data

5.1 INTRODUCTORY COMMENTS

Procedures to calculate the probabilities of events using many continuous and discrete random variables are discussed in Chapter 4. However, determining the underlying distribution of a random variable in the first place is a challenge to engineers. To define a distribution uniquely, its parameters need to be estimated. Generally, the distribution and its parameters are estimated using available data. Thus, it is important to study available procedures to determine the distribution and its parameters for a given set of data of a random variable. All subsequent evaluations of risk and reliability depend on these evaluations, which are the subject of this chapter.

5.2 DETERMINATION OF PROBABILITY DISTRIBUTION

In practice, the choice of probability distribution may be dictated by mathematical convenience or by familiarity with a distribution. In some cases, in the absence of any other information, the underlying distribution can be assumed to be uniform, triangular, trapezoidal, and so forth, as discussed in Section 4.6. When sufficient data are available, a histogram or frequency diagram can be used to determine the underlying distribution; however, more than one distribution may fit the available data, as shown in Figure 3.1 in Chapter 3. In some cases, the physical process may suggest a specific form of distribution.

As an example, Young's modulus is fequently modeled as a lognormal random variable in the literature. The task is to establish its validity, based on sample information such as that given in Table 3.1. The underlying distribution can be established in several ways, including (a) drawing a frequency diagram, (b) plotting the data on probability paper, and (c) conducting some statistical tests known as goodness-of-fit tests for

distribution. The concept behind the frequency diagram was discussed in detail in Section 3.1 in Chapter 3 and will not be repeated here. The uses of probability papers and statistical tests are discussed in the following sections.

5.2.1 Probability Papers

The underlying distribution can be established by plotting the available data for a random variable, such as that given in Table 3.1 in Chapter 3, and the corresponding cumulative probabilities on suitably constructed graph paper or *probability paper*. The scale of the graph paper should be such that when the random variables and the corresponding cumulative probabilities are plotted, a linear relationship between the two will be produced. It is known that a uniformly distributed random variable and its CDF have a linear relationship. Thus, if the random variable and its CDF are plotted using simple arithmetic scale on ordinary graph paper and the relationship appears to be linear, it must have a uniform distribution. Obviously, different probability papers are needed for different distributions.

Probability papers are commercially available for normal or lognormal distributions and for the Weibull distribution. If the probability paper corresponding to a particular distribution is not readily available, it can be constructed as discussed in Section 5.2.2. Assuming that the normal and lognormal probability papers are available, we will first discuss their use in determining the underlying distribution.

5.2.1.1 Plotting on Normal and Lognormal Probability Papers

Although there are several alternatives, the procedure proposed by Gumbel (1954) is discussed here because of its mathematical simplicity and wide acceptance. Suppose there are N observations, $x_1, x_2, ..., x_m, ..., x_N$, of a random variable. In the first step, the data need to be arranged in increasing order. In the second step, the mth value of the random variable is plotted at the CDF of $m/(N + 1)$ on the probability paper of interest. The procedure is further explained in Table 5.1 using the data on Young's modulus shown in Table 3.1.

The Young's modulus versus the corresponding CDF is plotted in Figure 5.1 on normal probability paper and in Figure 5.2 on lognormal probability paper. Although it is not necessary, the mean and the standard deviation or the COV of the Young's modulus can be estimated from the plots as shown in the figures. Comparison of Figures 5.1 and 5.2 shows that both the normal and lognormal plots appear to be straight lines; visual inspection does not establish the superiority of one over the other.

In general, frequency diagrams and probability papers are graphic or visual methods to determine the distribution of a random variable. More rigorous numerical tests are described in Section 5.2.3.

5.2.2 Construction of a Probability Paper

The probability paper can be constructed for a specific distribution using standard two-dimensional graph paper and changing its scale appropriately so that when the random variable and its CDF are plotted, the relationship appears to be a straight line. The

Table 5.1 Preparation of Data for Young's Modulus for Plotting on Probability Papers

m	E (ksi)	$m/(N+1)$	m	E (ksi)	$m/(N+1)$
1	25,900	1/42 = 0.0238	21	29,400	21/42 = 0.5000
2	27,400	0.0476	22	29,400	0.5238
3	27,400	0.0714	23	29,500	0.5476
4	27,500	0.0952	24	29,600	0.5714
5	27,600	0.1190	25	29,600	0.5952
6	28,100	0.1429	26	29,900	0.6190
7	28,300	0.1667	27	30,200	0.6429
8	28,300	0.1905	28	30,200	0.6667
9	28,400	0.2143	29	30,200	0.6905
10	28,400	0.2381	30	30,300	0.7143
11	28,700	0.2619	31	30,500	0.7381
12	28,800	0.2857	32	30,500	0.7619
13	28,900	0.3095	33	30,600	0.7857
14	29,000	0.3333	34	31,100	0.8095
15	29,200	0.3571	35	31,200	0.8333
16	29,300	0.3810	36	31,300	0.8571
17	29,300	0.4048	37	31,300	0.8810
18	29,300	0.4286	38	31,300	0.9048
19	29,300	0.4524	39	32,000	0.9286
20	29,300	0.4762	40	32,700	0.9524
			41	33,400	0.9762

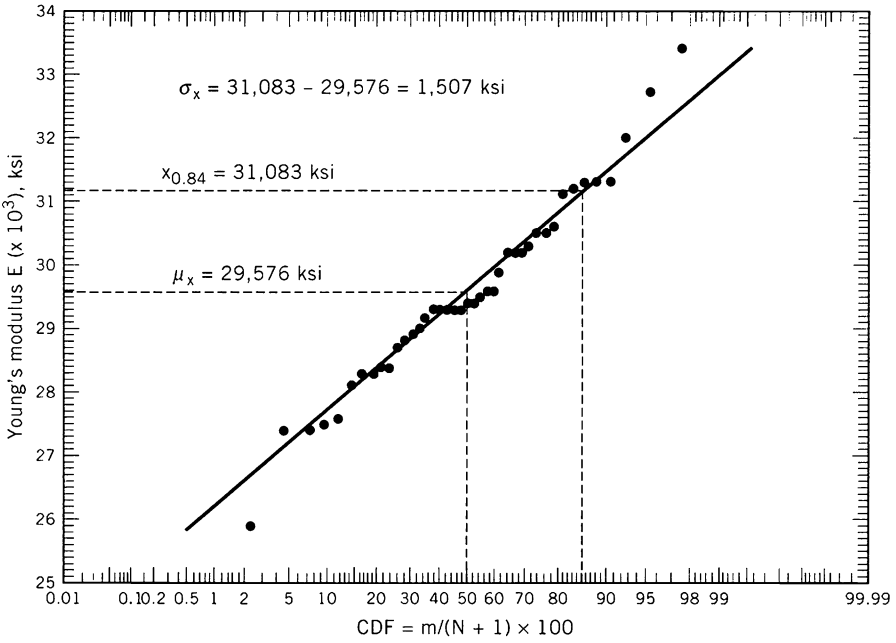


Figure 5.1 Normal Probability Paper

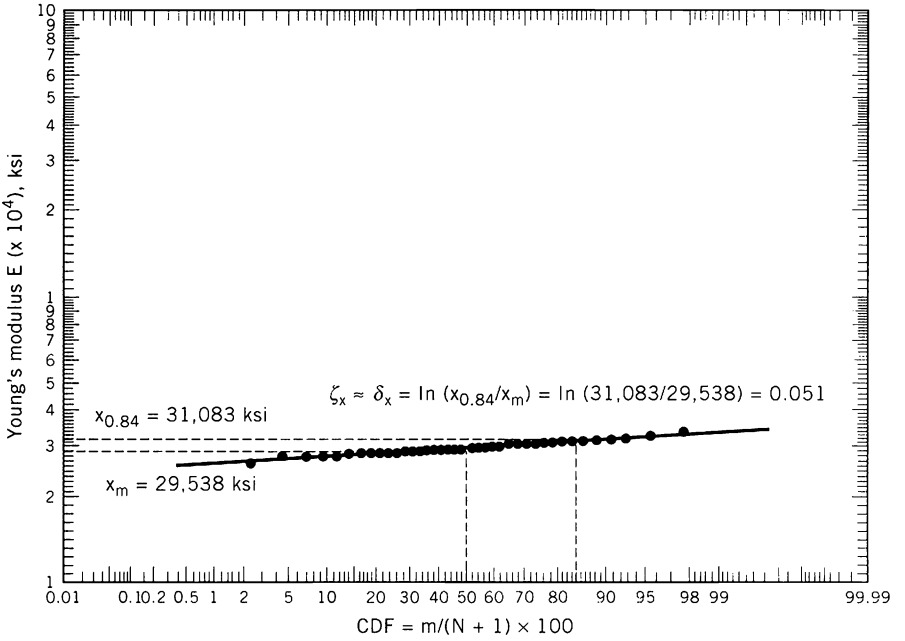


Figure 5.2 Lognormal Probability Paper

appropriate scale can be obtained by transforming the random variable into a *standard variate,* thereby making the CDF independent of the parameters of the distribution. For example, consider a random variable with a Rayleigh distribution. The PDF of the Rayleigh distribution can be shown to be

$$\begin{aligned} f_X(x) &= \frac{x}{\alpha^2} e^{-\frac{1}{2}\left(\frac{x}{\alpha}\right)^2}, \quad x \geq 0 \\ &= 0, \quad x < 0 \end{aligned} \tag{5.1}$$

where α is the parameter of the distribution. In fact, it can be shown that α is the modal value or the most probable value of the Rayleigh distribution. The standard variate, S, for the Rayleigh distribution is

$$S = X / \alpha. \tag{5.2}$$

With this variable transformation and using Equation 6.7, discussed in detail in Chapter 6, the PDF of S can be shown to be

$$\begin{aligned} f_S(s) &= s \cdot e^{-\frac{1}{2}s^2}, \quad s \geq 0 \\ &= 0, \quad s < 0. \end{aligned} \tag{5.3}$$

The corresponding CDF of S is

$$F_S(s) = 1 - e^{-\frac{s^2}{2}}. \tag{5.4}$$

Thus, the CDF of S has been rendered independent of α, the parameter of the Rayleigh distribution. Then, for a given value of $F_S(s)$, the corresponding value of s can be easily calculated as shown in Table 5.2.

Table 5.2 Data for Developing Rayleigh Probability Paper

s	$F_S(s)$	s	$F_S(s)$
0.46	0.10	2.25	0.92
0.67	0.20	2.31	0.93
0.84	0.30	2.37	0.94
1.01	0.40	2.45	0.95
1.18	0.50	2.49	0.955
1.26	0.55	2.54	0.96
1.35	0.60	2.59	0.965
1.45	0.65	2.65	0.97
1.55	0.70	2.72	0.975
1.67	0.75	2.80	0.98
1.79	0.80	2.90	0.985
1.95	0.85	3.03	0.99
2.15	0.90	3.26	0.995

In Figure 5.3, the standard variate S is plotted along the X-axis in a convenient arithmetic scale. Along the X-axis, but parallel to S, the $F_S(s)$ values are also plotted. No scale change is necessary for the Y-axis. The graph paper thus generated is the Rayleigh probability paper. If the relationship appears to be linear when any random variable X is plotted along the Y-axis in a suitable arithmetic scale, and the corresponding CDF values estimated to be $m/(N + 1)$ as in Table 5.1 are plotted along the X-axis following the $F_S(s)$ scale, then the underlying distribution can be considered to be Rayleigh. The essence of developing a probability paper for a given distribution is appropriately transforming the random variable into a standard variate. Once this is properly done, the CDF of the standard variate can be calculated easily and the information will help to identify the suitable scale along the X-axis. The scale along the Y-axis will remain arithmetic in most cases (except cases like the lognormal distribution), so no additional work is necessary. The graph thus generated is the probability paper representing the underlying distribution.

If the probability papers for the normal and lognormal distributions are not available from commercial sources, they can be easily generated using this procedure. For the normal distribution, the transformation to generate the standard variable is given by Equation 4.3. For the lognormal random variable with parameters λ and ζ, the required transformation is given by Equation 4.12.

EXAMPLE 5.1

The following example on Rayleigh distribution demonstrates the procedures just discussed. Fatigue is a very important design consideration for steel bridges. Estimation of fatigue loading on highway bridges caused by the passage of passenger cars, pickups, trailers, trucks, and other traffic is a major source of uncertainty. The load effect on bridges is usually modeled using the stress range parameter. It has been reported in the literature that the uncertainty in the stress range parameter of highway bridges can be modeled by the Rayleigh distribution. The following data on the stress range parameter P in ksi units is available for a particular bridge: 1.1, 3.7, 0.5, 1.3, 1.4, 0.8, 2.5, 2.7, 3.3, 1.0, 4.0, 0.7, 2.1, 2.0, 1.2, 3.0, 1.6, 1.9, 1.8, 2.3, 1.5, 2.8, 2.6, and 2.2.

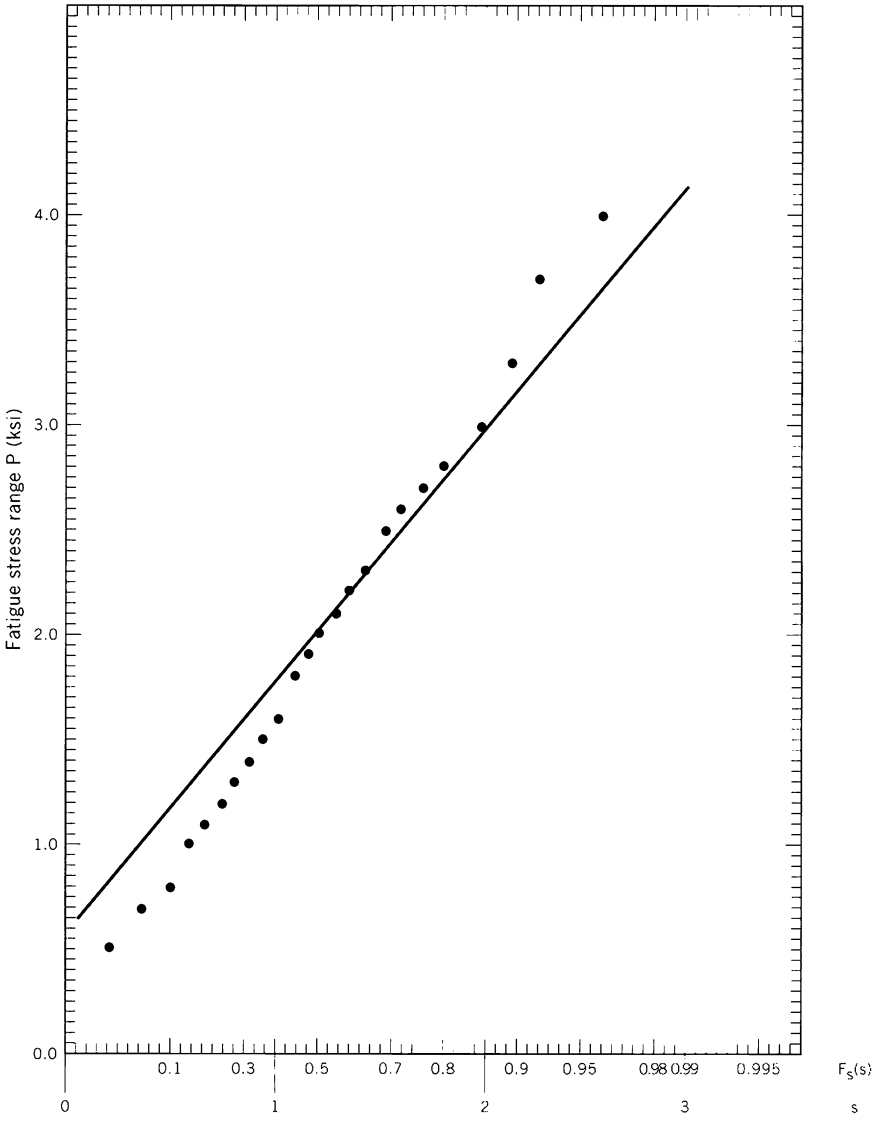


Figure 5.3 Construction of Rayleigh Probability Paper

First, construct a Rayleigh probability paper. Then, plot the data for P on it, and state whether the underlying distribution is Rayleigh.

SOLUTION

The data given in Table 5.2 can be used to construct a Rayleigh probability paper as shown in Figure 5.3. Then the data for P are arranged in increasing order as shown in Table 5.3. When the data are plotted, as shown in Figure 5.3, they appear to form a straight line. Thus, the distribution of the fatigue stress range parameter can be modeled as a Rayleigh distribution.

Table 5.3 Fatigue Stress Range Data

m	P (ksi)	$m/(N+1)$	m	P (ksi)	$m/(N+1)$
1	0.5	$1/(24+1) = 0.04$	13	2.0	$13/(24+1) = 0.52$
2	0.7	0.08	14	2.1	0.56
3	0.8	0.12	15	2.2	0.60
4	1.0	0.16	16	2.3	0.64
5	1.1	0.20	17	2.5	0.68
6	1.2	0.24	18	2.6	0.72
7	1.3	0.28	19	2.7	0.76
8	1.4	0.32	20	2.8	0.80
9	1.5	0.36	21	3.0	0.84
10	1.6	0.40	22	3.3	0.88
11	1.8	0.44	23	3.7	0.92
12	1.9	0.48	24	4.0	0.96

5.2.3 Statistical Tests

Determining the underlying distribution of a random variable using probability paper appears to be cumbersome, particularly when the probability paper needs to be constructed. Furthermore, even after the plotting, a judgment needs to be made as to whether the relationship between the random variable and its CDF is close to linear. A perfect linear relationship is very rarely obtained. More definitive and less cumbersome statistical tests of goodness-of-fit can be conducted to establish the underlying distribution. Two commonly used statistical tests for this purpose are the *chi-square* (χ^2) and the *Kolmogorov-Smirnov* (K-S) tests. The chi-square test is based on the error between the observed and assumed PDF of the distribution, and the K-S test is based on the error between the observed and assumed CDF of the distribution, as discussed in the following sections.

5.2.3.1 Chi-Square Test

In the χ^2 goodness-of-fit test, the range of the n observed data is divided into m intervals, and the number of times (n_i) the random variable is observed in the ith interval is counted ($i = 1$ to m). Observed frequencies $n_1, n_2, \ldots, n_m$ of m intervals of the random variable are then compared with the corresponding theoretical frequencies $e_1, e_2, \ldots, e_m$ of an assumed distribution. It can be shown (Hoel, 1962) that the quantity

$$\sum_{i=1}^{m} \frac{(n_i - e_i)^2}{e_i} \tag{5.5}$$

approaches the χ^2 distribution with $f = m - 1 - k$ degrees of freedom as the total sample points n tends to ∞. The χ^2 distribution is more formally presented in Section 5.4.6. Here, m is the number of intervals and k is the number of distribution parameters estimated from the data. The number of degrees of freedom f is a parameter of the χ^2 distribution. A significance level α is selected. Significance levels between 1% and 10% are common. A significance level of 5% implies that for 5 out of a total of 100 different samples, the assumed theoretical distribution cannot be an acceptable model. If $c_{1-\alpha,f}$ is denoted as the value of the χ^2 distribution with f degrees of freedom at CDF of $(1 - \alpha)$, and if the

value obtained from Equation 5.5 is less than this value, then the assumed distribution is acceptable at the significance level α. Thus, the assumed distribution will be acceptable at the significance level α if

$$\sum_{i=1}^{m} \frac{(n_i - e_i)^2}{e_i} < c_{1-\alpha,f}. \tag{5.6}$$

The outcome of the χ^2 statistical test depends on the number of intervals m used and the significance level α selected. Both need to be chosen based on judgment. $c_{1-\alpha,f}$ values for the χ^2 distribution are given in Appendix 3. It is desirable that both m and the e_i should be greater than or equal to 5 to obtain satisfactory results. However, this may not always be possible.

EXAMPLE 5.2

To determine whether the observations on Young's modulus given in Table 3.1 allow it to be accepted as a normal or lognormal random variable at the 5% significance level, the steps summarized in Table 5.4 can be followed.

All 41 observations on Young's modulus are divided into five intervals, and the number of observations in each interval is counted and tabulated under n_i. The theoretical frequency for each interval for normal and lognormal distributions is calculated. The first entry for theoretical frequency for normal distribution is 6.021. It can be calculated as

$$e_1 = P(E \le 28{,}000) \times 41$$
$$= \Phi\left(\frac{28{,}000 - 29{,}576}{1{,}507}\right) \times 41 = \Phi(-1.05) \times 41 = 0.14686 \times 41 = 6.021.$$

The theoretical frequency of the Young's modulus for the interval 28,000 to 29,000 ksi can be calculated as

$$e_2 = P(28{,}000 < E \le 29{,}000) \times 41$$
$$= \left[\Phi\left(\frac{29{,}000 - 29{,}576}{1{,}507}\right) - 0.14686\right] \times 41 = 8.410.$$

The theoretical frequencies for other intervals can similarly be calculated. The theoretical frequencies for all intervals must add up to the total number of observations (i.e., 41

Table 5.4 χ^2 Test on Young's Modulus for Golden Gate Bridge

Young's modulus (ksi)	Observed frequency n_i	Theoretical frequency e_i		$(n_i - e_i)^2/e_i$	
		Normal	Lognormal	Normal	Lognormal
< 28,000	5	6.021	6.116	0.173	0.204
28,000–29,000	8	8.410	8.774	0.020	0.068
29,000–30,000	13	10.590	10.600	0.548	0.543
30,000–31,000	7	8.966	8.600	0.431	0.298
> 31,000	8	7.013	6.910	0.139	0.172
Total	41	41	41	1.311	1.285

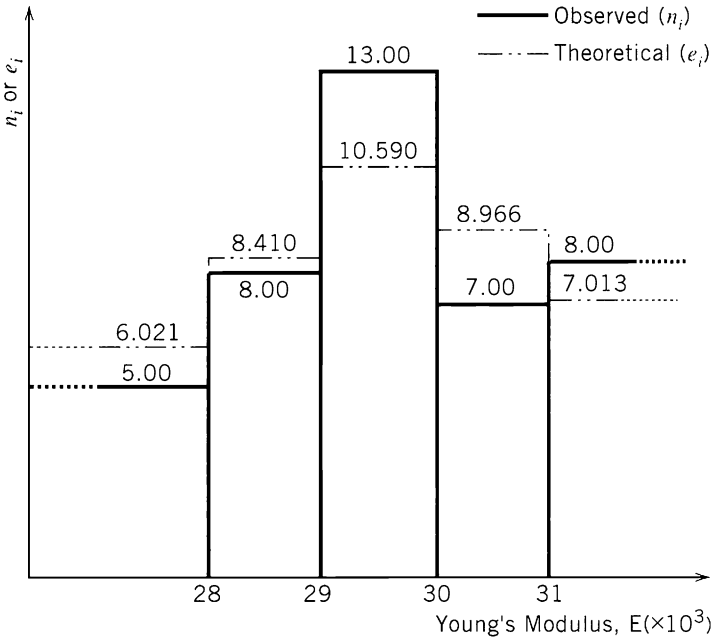


Figure 5.4 χ^2 Test on Young's Modulus for Normal Distribution

in this example, as shown in Table 5.4). The observed frequency in each interval can simply be counted from the observed data as noted in Table 5.4. As mentioned earlier, the intervals are selected in such a way that the theoretical frequencies are at least 5.

The observed and theoretical frequencies for normal distribution are plotted in Figure 5.4. Using Equation 5.5, we find that the total error for normal distribution is 1.311.

For both the normal and lognormal distributions, their two parameters are estimated from the sample mean and variance. Thus, for both distributions, the degrees of freedom are $f = 5 - 1 - 2 = 2$. For the significance level $\alpha = 5\%$, the corresponding $c_{0.95,2}$ from Appendix 3 is found to be 5.991, which is greater than 1.311. Thus, the normal distribution is acceptable with a 5% significance level.

The theoretical and observed frequencies for lognormal distribution can similarly be calculated as shown in Table 5.4. Using Equation 5.5, we can calculate the total error for the lognormal distribution to be 1.285. The error is less than 5.991; thus, the lognormal distribution is also acceptable with a 5% significance level. However, the lognormal distribution is slightly better than the normal distribution.

5.2.3.2 Kolmogorov-Smirnov (K-S) Test

The K-S test compares the observed cumulative frequency and the CDF of an assumed theoretical distribution. The first step is to arrange the data in increasing order. Then the maximum difference between the two cumulative distribution functions of the ordered data can be estimated as

$$D_n = \max | F_X(x_i) - S_n(x_i) | \tag{5.7}$$

where $F_X(x_i)$ is the theoretical CDF of the assumed distribution at the ith observation of the ordered samples x_i, and $S_n(x_i)$ is the corresponding stepwise CDF of the observed ordered samples. $S_n(x_i)$ can be estimated as

$$S_n(x_i) = \begin{Bmatrix} 0 \ , & x < x_1 \\ \dfrac{m}{n} \ , & x_m \le x \le x_{m+1} \\ 1 \ , & x \ge x_n \end{Bmatrix}. \tag{5.8}$$

The concept is shown in Figure 5.5. Mathematically, D_n is a random variable and its distribution depends on the sample size n. The CDF of D_n can be related to the significance level α as

$$P(D_n \le D_n^{\alpha}) = 1 - \alpha \tag{5.9}$$

and the $D_n{}^{\alpha}$ values at various significance levels α can be obtained from a standard mathematical table as shown in Appendix 4. Then, according to the K-S test, if the maximum difference D_n is less than or equal to the tabulated value $D_n{}^{\alpha}$, the assumed distribution is acceptable at the significance level α.

The advantage of the K-S test over the χ^2 test is that it is not necessary to divide the data into intervals; thus the error or judgment associated with the number or size of the interval is avoided.

EXAMPLE 5.3

The same Young's modulus data given in Table 3.1 are considered. For a normal distribution, the two parameters are μ_E = 29,576 ksi and σ_E = 1,507 ksi. For a lognormal distribution, the two parameters are λ_E = 10.293 and ζ_E = 0.051.

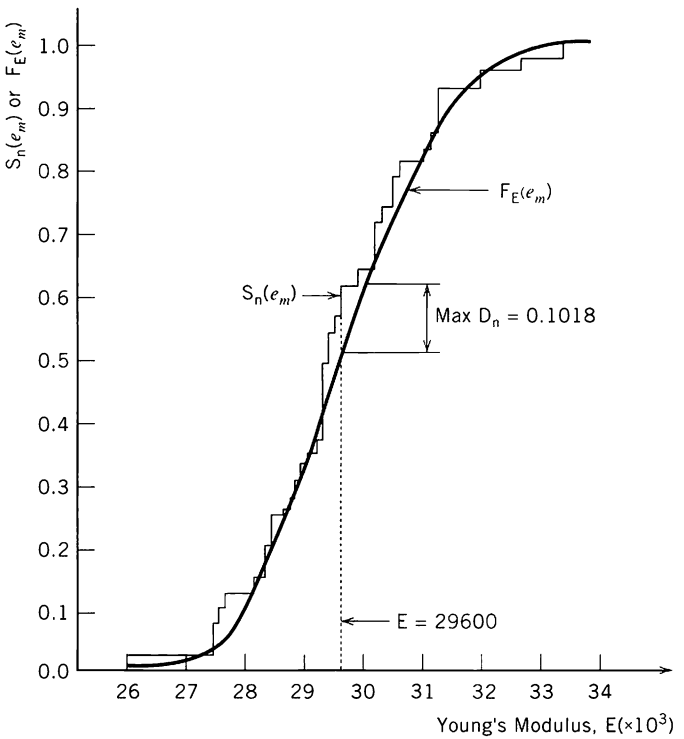


Figure 5.5 K-S Test on Young's Modulus for Normal Distribution

Table 5.5 K-S Test on Young's Modulus for the Golden Gate Bridge

m	E(ksi)	$S_n(e_m) = m/n$	$F_E(e_m)$		$D_n = \lvert F_E(e_m) - S_n(e_m) \rvert$	
			Normal	Lognormal	Normal	Lognormal
1	25,900	0.0244	0.0073	0.0051	0.0171	0.0193
2	27,400	0.0488	0.0749	0.0721	0.0261	0.0233
3	27,400	0.0732	0.0749	0.0721	0.0017	0.0011
4	27,500	0.0976	0.0838	0.0823	0.0138	0.0153
5	27,600	0.1220	0.0951	0.0934	0.0269	0.0286
6	28,100	0.1463	0.1635	0.1660	0.0172	0.0197
7	28,300	0.1707	0.1977	0.2033	0.0270	0.0326
8	28,300	0.1951	0.1977	0.2033	0.0026	0.0082
9	28,400	0.2195	0.2177	0.2236	0.0018	0.0041
10	28,400	0.2439	0.2177	0.2236	0.0262	0.0203
11	28,700	0.2683	0.2810	0.2877	0.0127	0.0194
12	28,800	0.2927	0.3050	0.3121	0.0123	0.0194
13	28,900	0.3171	0.3264	0.3372	0.0093	0.0201
14	29,000	0.3415	0.3520	0.3632	0.0105	0.0217
15	29,200	0.3659	0.4013	0.4129	0.0354	0.0470
16	29,300	0.3902	0.4286	0.4404	0.0384	0.0502
17	29,300	0.4146	0.4286	0.4404	0.0140	0.0258
18	29,300	0.4390	0.4286	0.4404	0.0104	0.0014
19	29,300	0.4634	0.4286	0.4404	0.0348	0.0230
20	29,300	0.4878	0.4286	0.4404	0.0592	0.0474
21	29,400	0.5122	0.4522	0.4681	0.0600	0.0441
22	29,400	0.5366	0.4522	0.4681	0.0844	0.0685
23	29,500	0.5610	0.4801	0.4920	0.0809	0.0690
24	29,600	0.5854	0.5080	0.5199	0.0774	0.0655
25	29,600	0.6098	0.5080	0.5199	**0.1018**	**0.0899**
26	29,900	0.6341	0.5871	0.5987	0.0470	0.0354
27	30,200	0.6585	0.6591	0.6700	0.0006	0.0115
28	30,200	0.6829	0.6591	0.6700	0.0238	0.0129
29	30,200	0.7073	0.6591	0.6700	0.0482	0.0373
30	30,300	0.7317	0.6844	0.6950	0.0473	0.0367
31	30,500	0.7561	0.7291	0.7389	0.0270	0.0172
32	30,500	0.7805	0.7291	0.7389	0.0514	0.0416
33	30,600	0.8049	0.7517	0.7580	0.0532	0.0469
34	31,100	0.8293	0.8438	0.8461	0.0145	0.0168
35	31,200	0.8537	0.8599	0.8599	0.0062	0.0062
36	31,300	0.8780	0.8729	0.8729	0.0051	0.0051
37	31,300	0.9024	0.8729	0.8729	0.0295	0.0295
38	31,300	0.9268	0.8729	0.8729	0.0539	0.0539
39	32,000	0.9512	0.9463	0.9429	0.0049	0.0083
40	32,700	0.9756	0.9808	0.9773	0.0052	0.0017
41	33,400	1.0000	0.9945	0.9922	0.0055	0.0078

The maximum differences D_n for the normal and lognormal distributions are calculated as shown in Table 5.5 and are found to be 0.1018 and 0.0899, respectively. The results for the normal distribution are also plotted in Figure 5.5. For a 5% significance level and 41 sample points, $D_{41}{}^{0.05}$ is found to be 0.21 from Appendix 4. Thus, both normal and lognormal distributions are acceptable with a 5% significance level for the K-S test. However, as observed in the χ^2 test, the lognormal distribution is better than the normal distribution.

5.3 ESTIMATION OF PARAMETERS OF A DISTRIBUTION

Once the type of distribution of a random variable is selected, it is necessary to define it uniquely by evaluating its parameters. Some distributions, such as the binomial and Poisson, have only one parameter. Others, such as the normal and lognormal, have two parameters, and others could have more than two parameters. The accuracy in estimating these parameters based on the test or observational data determines the success in modeling the uncertainty in a random variable.

Once the randomness is uniquely defined in terms of the parameters of a distribution, it is used in subsequent probabilistic analyses, assuming the basic characteristics of the random variable remain unchanged. In other words, adding more samples to the original database is not expected to change the distribution characteristics. In the literature, this is generally known as the *point estimation* of parameters. Point estimation plays a very important role in probabilistic evaluation, and certain properties are desirable in a point estimator: *unbiasedness, consistency, efficiency,* and *sufficiency.* Unbiasedness means the expected value or mean of the estimator is the parameter itself. Consistency means that as the sample size approaches infinity, the estimator approaches the value of the parameter. Efficiency refers to the variance of the estimator; less variance is always preferable. Sufficiency refers to the ability of an estimator to extract all the pertinent information from a sample to determine the parameter. Equation 3.2 represents the unbiased estimation of the variance by using a denominator of $(N - 1)$, instead of simply N.

For a given sample, the two most commonly used methods of point estimation of parameters are the *method of moments* and the *method of maximum likelihood.*

5.3.1 Method of Moments

As discussed earlier, the mean or expected value of a sample represents its first moment, the variance represents the second moment, the skewness represents the third moment, and so on. The basic concept behind the method of moments is that all the parameters of a distribution can be estimated using the information on its moments. Thus, if a distribution has a single parameter, then only one piece of information needs to be extracted from the sample; most likely, this will be the first moment, or the mean value of the random variable. If a distribution has two parameters, then two pieces of information need to be extracted from the sample, and most likely, they will be the first two moments, that is, the mean and the variance of the random variable. As the name of the method implies, the parameters of a distribution have a definite relationship with the moments of the random variable. The relationships are tabulated for several commonly used distributions in Table 5.6.

Table 5.6 Estimation of Parameters Using Method of Moments

Distribution	Probability density function (PDF) or mass function (PMF)	Parameters	Relation to mean and variance
Continuous random variables			
Normal	$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]$ $-\infty < x < \infty$	μ_X, σ_X	$E(X) = \mu_X$ $\mathrm{Var}(X) = \sigma_X^2$
Lognormal	$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right]$ $x \geq 0$	λ, ζ	$E(X) = \exp\left(\lambda + \frac{1}{2}\zeta^2\right)$ $\mathrm{Var}(X) = E^2(X)[e^{\zeta^2} - 1]$
Beta	$f_X(x) = \frac{1}{B(q,r)} \frac{(x-a)^{q-1}(b-x)^{r-1}}{(b-a)^{q+r-1}}$ $a \leq x \leq b$	a, b, q, r	$E(X) = a + \frac{q}{q+r}(b-a)$ $\mathrm{Var}(X) = \frac{qr}{(q+r)^2(q+r+1)}(b-a)^2$
Rayleigh	$f_X(x) = \frac{x}{\alpha^2} \exp\left[-\frac{1}{2}\left(\frac{x}{\alpha}\right)^2\right]$ $x \geq 0$	α	$E(X) = \sqrt{\frac{\pi}{2}}\alpha$ $\mathrm{Var}(X) = \left(2 - \frac{\pi}{2}\right)\alpha^2$
Exponential	$f_X(x) = \nu e^{-\nu x}$ $x \geq 0$	ν	$E(X) = \frac{1}{\nu}$ $\mathrm{Var}(X) = \frac{1}{\nu^2}$
Discrete random variables			
Binomial	$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, 2, \ldots, n$	p	$E(X) = np$ $\mathrm{Var}(X) = np(1-p)$
Geometric	$p_X(x) = p(1-p)^{x-1}$ $x = 0, 1, 2, \ldots$	p	$E(X) = 1/p$ $\mathrm{Var}(X) = (1-p)/p^2$
Poisson	$p_X(x) = \frac{(\nu t)^x}{x!} e^{-\nu t}$ $x = 0, 1, 2, \ldots$	ν	$E(X) = \nu t$ $\mathrm{Var}(X) = \nu t$

The normal distribution has two parameters, and they are essentially the first two moments of the random variable. The lognormal distribution also has two parameters; however, they can be estimated from the first two moments through two formulas, as discussed in Section 4.2.2. In some cases, the moments are the parameters, and in other cases, the parameters can be estimated from the moments.

5.3.2 Method of Maximum Likelihood

The principle behind the maximum likelihood method is that for a random variable X, if $x_1, x_2, \ldots, x_n$ are the n observations or sample values, then the estimated value of the parameter is the value most likely to produce these observed values. Consider the den-

sity function of X to be $f_X(x, p)$, where p is the parameter that needs to be estimated. In random sampling, the x_i's are assumed to be independent; if the likelihood of observing x_i's is proportional to their corresponding density functions, the *likelihood function* can be defined as

$$L(x_1, x_2, \ldots, x_n; p) = f_X(x_1, p) f_X(x_2, p) \ldots, f_X(x_n, p). \tag{5.10}$$

Maximizing the likelihood function with respect to the parameter, that is, equating $\partial L/\partial p$ to zero, will give the equation required to estimate the value of p according to the maximum likelihood method. If m parameters are required to describe a distribution, they can be estimated by solving m simultaneous equations obtained by taking the derivatives of the likelihood function with respect to each of them.

For mathematical simplicity and observing that the likelihood function is multiplicative in nature, using the logarithm of the likelihood function is much simpler, as shown in the following examples.

EXAMPLE 5.4

Geometric Distribution

Assume that the random variable X has a geometric distribution and n observations are available. As shown in Table 5.6, the geometric distribution has only one parameter p. The PMF of the geometric distribution is

$$p_X(x) = p(1-p)^{x-1}.$$

Equation 5.10 shows that the likelihood function of the observed data is

$$L(x_1, x_2, \ldots, x_n; p) = \prod_{i=1}^{n} p(1-p)^{x_i - 1}.$$

Taking the logarithm of both sides,

$$\ln L = n \ln p + \sum_{i=1}^{n} (x_i - 1) \ln(1-p)$$

$$= n \ln p - n \ln(1-p) + \ln(1-p) \sum_{i=1}^{n} x_i$$

or

$$\frac{\partial \ln L}{\partial p} = n \frac{1}{p} + n \cdot \frac{1}{1-p} - \frac{1}{1-p} \cdot \sum_{i=1}^{n} x_i = 0.$$

After simplifying this equation, it can be shown that the estimated value of p, denoted as $\hat{p}$, is

$$\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{E(X)}$$

Normal Distribution

The likelihood function of n observations of a normal random variable X is

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{x_i - \mu_X}{\sigma_X}\right)^2}.$$

Hence,

$$\ln L = -n \ln \sqrt{2\pi} - n \ln \sigma_X - \frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i - \mu_X}{\sigma_X} \right)^2.$$

A derivative of the logarithm of the likelihood function with respect to the mean μ_X and the standard deviation σ_X will give the estimated value of the corresponding parameters, denoted as $\bar{x}$ and s_X^2, respectively, as shown below.

$$\frac{\partial \ln L}{\partial \mu_X} = \frac{1}{2} \sum_{i=1}^{n} 2 \left(\frac{x_i - \bar{x}}{s_X^2} \right) = 0$$

or

$$\sum_{i=1}^{n} (x_i - \bar{x}) = 0$$

or

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

Similarly,

$$\frac{\partial \ln L}{\partial \sigma_X} = -\frac{n}{s_X} + \frac{1}{2} \sum_{i=1}^{n} (x_i - \bar{x})^2 \frac{2}{s_X^3} = 0$$

or

$$s_X^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}.$$

In both examples, the method of moments and the maximum likelihood method essentially give the same estimate of the parameters.

5.4 INTERVAL ESTIMATION OF MEAN AND VARIANCE

Point estimation of parameters using the method of moments as discussed in the previous section requires information on mean and variance. Samples are always used to estimate the mean and variance, leading to the estimation of the parameters of a distribution. But how good is the estimation of mean and variance? To answer this important question, two additional terms, *population* and *sample*, need to be introduced. A population represents all conceivable observations of a random variable. A population could be the Young's modulus for a particular grade of steel, or the wind velocity recorded following standard procedures all over the world, or the opinion on an issue of everyone in the U.S. However, it may be impractical to collect information on an entire population. To circumvent this problem, representative samples are taken from the population to define the population characteristics. In opinion polls, it is very common to contact 1,000 to 1,500 scientifically selected persons in the U.S. to represent the total population of over 250 million. The quality of information collected from a sample will depend on the size of the sample, the underlying uncertainty in the random variable under consideration, and the confidence level required in making a conclusion.

Since it is impractical to collect information from all the available sources, the information on the sample mean and variance is used to estimate the population mean and variance. This introduces some error in the prediction of the population mean and vari-

ance, which are unknown but have constant values. A range of values in which they may be located is referred to as a *confidence interval*. In essence, the interval estimation method calculates the confidence intervals of the mean and variance.

In some cases, for example, opinion polls, the spread above and below the predicted value (known in casual presentation as the plus-minus values about the prediction) is of interest. This is known as the *two-sided confidence interval*. However, in some cases, the *one-sided confidence limit* might be of interest. The one-sided confidence limit can be the *lower confidence limit* or *upper confidence limit*.

Consider the compressive strength of concrete, which is a resistance-related parameter. For a project, the concrete will be acceptable if its strength is at least 4,000 psi. A few sample cylinders may be tested to estimate the mean and variance of compressive strength. If information from small samples is used, it may be of interest to know if the population mean is at least 4,000 psi for a required level of confidence. This value is known as the lower confidence limit. In general, for resistance-related random variables, a lower confidence limit of mean is often of interest. On the other hand, for the load-related random variables (e.g., wind speed) obtained using sampling information, it may be of interest to know if the population mean is less than a specific value for a required level of confidence. This value is known as the upper confidence limit. In a much broader sense, for example, for air or water pollution problems, emissions of radioactivity from a nuclear power plant, opinion polls, or vehicle speed on roadways, the lower or upper confidence limit of mean and variance may be of interest depending upon the problem.

In some cases, concerned parties (e.g., the supplier of concrete or the operator of a measuring device) may have an experience-based idea of the amount of uncertainty that can be expected in terms of variance, standard deviation, or COV. Thus, in some problems the information on the variance or standard deviation of a population may be known in advance, and in other cases it may need to be estimated from the collected samples. The two-sided confidence interval, or lower or upper confidence limit of the mean and variance of a population, needs to be estimated considering the sample size, the confidence level required for the prediction, and the amount of uncertainty (known or unknown) present in the random variable under consideration. They are discussed in the following sections.

5.4.1 Interval Estimation for the Mean with Known Variance

Suppose X is a random variable whose variance σ^2 is known. A random sample of size n (i.e., $x_1, x_2, ..., x_n$) is collected, and the sample mean $\bar{x}$ is calculated using Equation 3.1. The question is, how good is the sample mean obtained from a sample of size n in predicting the population mean μ, which is unknown but a constant. If another sample of size n is collected, the sample mean is expected to be somewhat different. In fact, each one of the observations $x_1, x_2, ..., x_n$ can be considered to have come from a set of independent random variables $X_1, X_2, ..., X_n$. All the X_i's have the same distribution as X. In this context, the sample mean is itself a random variable; since it is a random variable, it can be denoted as $\bar{X}$ (by an uppercase letter, as discussed in Chapter 3), and can be estimated as

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i. \tag{5.11}$$

The mean or expected value of the sample mean can be calculated as

$$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n} n\mu = \mu. \tag{5.12}$$

The expected value of the sample mean is equal to the population mean, indicating it is an unbiased estimator of the population mean μ.

The variance of the sample mean can be calculated as

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}. \tag{5.13}$$

The derivation of Equation 5.13 will be discussed in more detail in Chapter 6. Thus, the sample mean $\bar{X}$ has a mean of μ and a standard deviation of $(\sigma/\sqrt{n})$. If the underlying distribution of X is normal, then the sum of independent normal random variables is also a normal random variable; this will be discussed in detail in Chapter 6, Section 6.3.1.2. If the X_i's are independent normal variables, then Equation 5.11 indicates that the sample mean $\bar{X}$ will also have a normal distribution. When the sample size is relatively large, for instance greater than 30, then, by virtue of the central limit theorem (to be discussed in Section 6.3.1.5), the sample mean can be considered to have a normal distribution regardless of the distribution of X.

In summary, if the underlying distribution of the population is normal or if the sample size is relatively large, then the sample mean is a normal random variable, that is, $N(\mu, \sigma/\sqrt{n})$. With a simple transformation, as shown in Section 4.2.1 and using Equation 4.3, we can show that $(\bar{X} - \mu)/(\sigma/\sqrt{n})$ is a standard normal variate with zero mean and unit standard deviation. The probability that this standard normal variate falls between a specified lower and upper limit, usually selected symmetrically (i.e., $-k_{\alpha/2}$ and $k_{\alpha/2}$ as shown in Figure 5.6), can be written as

$$P\left(-k_{\alpha/2} \le \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \le k_{\alpha/2}\right) = 1 - \alpha \tag{5.14}$$

where $(1 - \alpha)$ is generally known as the confidence level, and $\pm k_{\alpha/2}$ are the values of the standard normal variate evaluated at the probability levels of $(1 - \alpha/2)$ and $\alpha/2$,

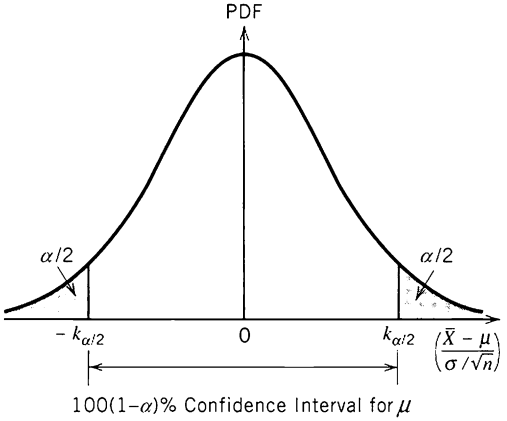


Figure 5.6 Two-Sided Confidence Interval for μ

respectively. In this equation, since a sample of size n is collected with a sample mean of $\bar{x}$, it gives a specific realization of the random variable $\bar{X}$, and thus, in Equation 5.14 $\bar{X}$ needs to be replaced by $\bar{x}$. Equation 5.14 can be simplified as

$$P\left(\bar{x} - k_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + k_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha \tag{5.15}$$

Equation 5.15 gives the two-sided $(1 - \alpha)$ confidence interval of the population mean μ. It can be interpreted as meaning that on the basis of a single sample of size n, there is confidence of $(1 - \alpha)$ that the estimated interval contains the population mean μ. Mathematically, the $(1 - \alpha)$ confidence interval for the mean can be simply stated as

$$<\mu>_{1-\alpha} = \left[\bar{x} - k_{\alpha/2}\frac{\sigma}{\sqrt{n}};\quad \bar{x} + k_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]. \tag{5.16}$$

EXAMPLE 5.5

Consider the 41 observations of the Young's modulus given in Table 5.1. As discussed earlier, the mean and standard deviation of the samples are 29,576 ksi and 1,507 ksi, respectively. Assume further that the Young's modulus is known to have a standard deviation of 1,507 ksi. Determine (a) the 95% confidence interval for the mean and (b) the 99% confidence interval for the mean.

SOLUTION

(a) For systematic evaluation of the confidence interval, the following steps should be followed.

Step 1

$$1 - \alpha = 0.95, \text{ or } \alpha = 1 - 0.95 = 0.05$$
$$\alpha / 2 = 0.05 / 2 = 0.025, \text{ and } 1 - \alpha / 2 = 1 - 0.025 = 0.975.$$

Step 2

Using the standard normal table given in Appendix 1, it can be shown that

$$k_{\alpha/2} = k_{0.025} = \Phi^{-1}(0.975) = 1.96.$$

Step 3

$$\frac{\sigma}{\sqrt{n}}k_{\alpha/2} = \frac{1,507}{\sqrt{41}}1.96 = 461.$$

According to Equation 5.16, the 95% confidence interval for the mean of the Young's modulus is

$$<\mu>_{0.95} = (29,576 - 461;\ 29,576 + 461) = (29,115;\ 30,037) \text{ ksi.}$$

Thus, based on the available information of 41 samples, there is a confidence level of 95% that the population mean for the Young's modulus is contained in the interval (29,115; 30,037) ksi, or the 95% confidence interval for the

Young's modulus is (29,115; 30,037) ksi.

(b) For the 99% confidence interval, the three steps just discussed can be followed again. In this case,

$$\alpha = 1 - 0.99 = 0.01, \quad \alpha/2 = 0.005, \quad \text{and} \quad 1 - \alpha/2 = 0.995$$

$$k_{0.005} = \Phi^{-1}(0.995) = 2.58$$

and

$$\frac{\sigma}{\sqrt{n}} k_{\alpha/2} = \frac{1,507}{\sqrt{41}} 2.58 = 607 \text{ ksi.}$$

Thus, using Equation 5.16, we find the 99% confidence interval for the mean to be

$$<\mu>_{0.99} = (29,576 - 607;\ 29,576 + 607) = (28,969;\ 30,183) \text{ ksi.}$$

EXAMPLE 5.6

Suppose another sample of size 10 is collected on the Young's modulus, and it has the same mean (i.e., 29,576 ksi) as the previous example and a known standard deviation of 1,507 ksi. (a) Determine the 95% confidence interval for the mean. (b) If the sample size is increased to 300 giving the same mean of 29,576 ksi, what would be the 95% confidence interval for the mean?

SOLUTION

(a) Using the information from the previous example, we can show that $k_{\alpha/2} = 1.96$, and

$$\frac{\sigma}{\sqrt{n}} k_{0.025} = \frac{1,507}{\sqrt{10}} 1.96 = 934 \text{ ksi.}$$

Thus, according to Equation 5.16, the 95% confidence interval for the mean is

$$<\mu>_{0.95} = (29,576 - 934;\ 29,576 + 934) = (28,642;\ 30,510) \text{ ksi.}$$

(b) For the sample size of 300, it can be shown that

$$\frac{\sigma}{\sqrt{n}} k_{\alpha/2} = \frac{1,507}{\sqrt{300}} 1.96 = 171 \text{ ksi.}$$

Thus,

$$<\mu>_{0.95} = (29,576 - 171;\ 29,576 + 171) = (29,405;\ 29,747) \text{ ksi.}$$

Several important observations can be made from the previous two examples. As the level of confidence increases, the confidence interval becomes wider. This is expected since a wider interval is more likely to contain the population mean μ, increasing the confidence in the prediction. Also, as the sample size increases, the confidence interval gets narrower for the same confidence level, indicating that a prediction using a larger sample size is always better. In fact, as the sample size approaches infinity, the sample mean will approach the population mean.

5.4.2 Lower and Upper Confidence Limit for the Mean with Known Variance

The previous section's discussion on the two-sided confidence interval is applicable for the lower or upper confidence limit for the mean; however, the limits of integration need to be changed as shown in Figure 5.7. Consider the lower confidence limit, $\mu_{>1-\alpha}$, first. In this case, referring to Figure 5.7a,

$$P\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le k_\alpha\right) = 1-\alpha \tag{5.17}$$

where $(1-\alpha)$ is the required confidence level and $k_\alpha = \Phi^{-1}(1-\alpha)$. For a specific sample mean of $\bar{x}$, Equation 5.17 can be rewritten as

$$P\left(\mu \ge \bar{x} - k_\alpha \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha. \tag{5.18}$$

Thus, the $(1-\alpha)$ lower confidence limit for the population mean μ is

$$\mu_{>1-\alpha} = \left(\bar{x} - k_\alpha \frac{\sigma}{\sqrt{n}}\right). \tag{5.19}$$

Similarly, referring to Figure 5.7b, the $(1-\alpha)$ upper confidence limit for the mean, $<\mu_{1-\alpha}$, can be shown to be

$$<\mu_{1-\alpha} = \left(\bar{x} + k_\alpha \frac{\sigma}{\sqrt{n}}\right). \tag{5.20}$$

The same three steps used to calculate the two-sided confidence interval can be used to calculate the lower and upper confidence limits, as discussed next.

EXAMPLE 5.7

Consider again the Young's modulus problem with 41 sample points. Determine (a) the 95% lower confidence limit for the mean, and (b) the 95% upper confidence limit for the mean.

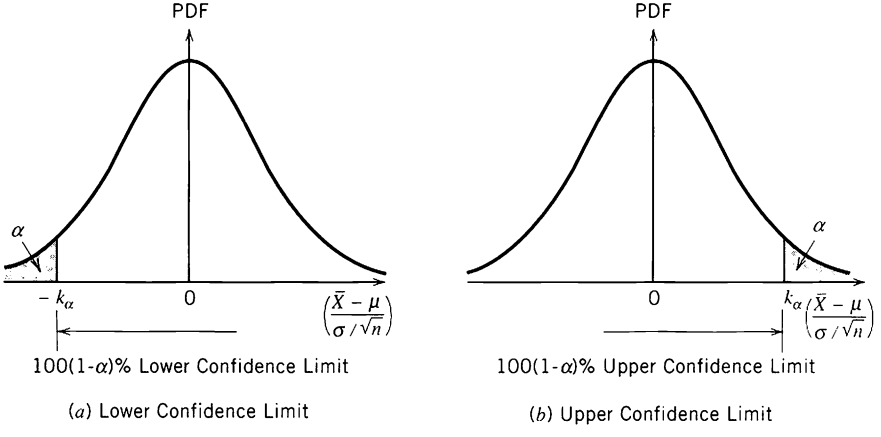


Figure 5.7 Lower and Upper Confidence Limits for μ

SOLUTION

(a) $1 - \alpha = 0.95$, or $\alpha = 0.05$
$k_{0.05} = \Phi^{-1}(0.95) = 1.65$
Thus,

$$k_\alpha \frac{\sigma}{\sqrt{n}} = 1.65 \frac{1,507}{\sqrt{41}} = 388 \text{ ksi.}$$

Using Equation 5.19, we find the 95% lower confidence limit for the mean to be

$$\mu >_{1-\alpha} = \left(\bar{x} - k_\alpha \frac{\sigma}{\sqrt{n}} \right) = 29,576 - 388 = 29,188 \text{ ksi.}$$

Simply stated, there is 95% confidence that the mean of the Young's modulus is at least 29,188 ksi.

(b) In this case, the first two steps in Part (a) of this example will remain the same. Using Equation 5.20, we can calculate the 95% upper confidence limit for the mean as

$$< \mu_{1-\alpha} = \left(\bar{x} + k_\alpha \frac{\sigma}{\sqrt{n}} \right) = 29,576 + 388 = 29,964 \text{ ksi.}$$

In other words, there is 95% confidence that the mean value of the Young's modulus is less than 29,964 ksi.

5.4.3 Interval Estimation for the Mean with Unknown Variance

The discussions in Sections 5.4.1 and 5.4.2 assume that the population variance is known: in other words, there is prior information on the population variance σ^2. In general, the population variance is not known in advance but needs to be estimated from the collected samples of size n. When the sample size n is relatively large, for instance, greater than 30, it can be assumed that the sample variance s^2 is a good estimator of the population variance σ^2. Thus, all the previous equations for estimating the confidence interval, and the lower and upper confidence limits (Equations 5.16, 5.19, and 5.20) are still applicable, except that σ should be replaced by s. However, if the sample size n is relatively small, for instance, less than 10, the assumption that the sample variance is equal to the population variance is not appropriate. In fact, common sense dictates that the confidence interval will be wider for smaller sample sizes, indicating more uncertainty in the prediction.

If the population variance cannot be assumed to be known, then the approach discussed earlier must be modified. If there is no prior information on the population variance, an exact confidence interval for the mean can be estimated if the underlying population is normal. For a nonnormal population distribution, if the sample size is relatively large, the error due to the normality assumption is expected to be relatively minor. If we assume that the population of X has a normal distribution and we denote $\bar{X}$ and S^2 as the sample mean and the variance, respectively, the probability distribution of $(\bar{X} - \mu) / (\sigma / \sqrt{n})$ is not normal. It can be shown that this standardized random variable has the t-distribution (or Student's t-distribution) with $(n - 1)$

degrees of freedom. William S. Gossett proposed the distribution under the pseudonym of "Student." The probability density function (PDF) of Student's t-distribution is given by

$$f_T(t) = \frac{\Gamma\left[(f+1)/2\right]}{\sqrt{\pi f}\,\Gamma(f/2)}\left(1+\frac{t^2}{f}\right)^{-(f+1)/2}, \quad -\infty < t < \infty \tag{5.21}$$

where $f = (n-1)$ is known as the *degrees of freedom*. The PDFs of a family of Student's t-distribution are shown in Figure 5.8 along with the standard normal variate. It can be observed that the PDFs of Student's t-distribution have symmetric bell-shaped curves similar to the normal curve. As f increases, the PDF of Student's t-distribution approaches the standard normal curve. When f is smaller, the PDF of Student's t-distribution is flatter than the PDF of the standard normal curve.

As with Section 5.4.1 and Equation 5.14, to calculate the $(1-\alpha)$ confidence interval for the mean with unknown variance, observing that the random variable $(\bar{X}-\mu)/(\sigma/\sqrt{n})$ has Student's t-distribution, we can make the following probability statement:

$$P\left(-t_{\alpha/2,n-1} \le \frac{\bar{X}-\mu}{S/\sqrt{n}} \le t_{\alpha/2,n-1}\right) = 1-\alpha. \tag{5.22}$$

Thus, for a sample of size n, with a sample mean of $\bar{x}$ and a standard deviation of s, the $(1-\alpha)$ confidence interval is

$$<\mu>_{1-\alpha} = \left[\bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}};\ \ \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}\right] \tag{5.23}$$

where $\pm t_{\alpha/2,\,n-1}$ are the values of Student's t-distribution with $(n-1)$ degrees of freedom evaluated at probabilities of $(1-\alpha/2)$ and $\alpha/2$, respectively. Equations 5.16 and 5.23 are very similar except that in Equation 5.16, the values of the standard normal variate are calculated at $(1-\alpha/2)$ and $\alpha/2$, and in Equation 5.23 the corresponding values are calculated for Student's t-distribution. For ease of estimation, the CDFs and the corresponding values of Student's t-distribution are tabulated in Appendix 5 for different degrees of freedom.

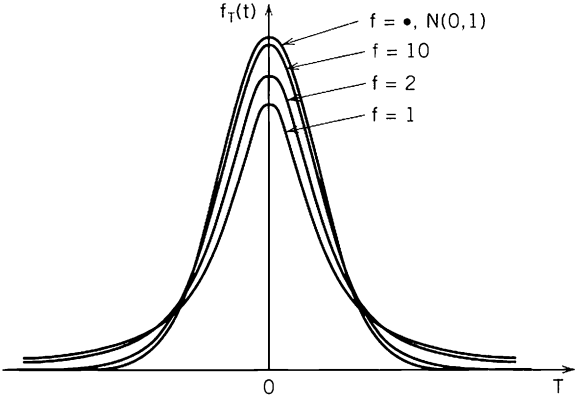


Figure 5.8 Student's t-distribution

EXAMPLE 5.8

Consider again Example 5.5a on Young's modulus. The sample mean and the standard deviation are calculated to be 29,576 and 1,507 ksi, respectively. Determine the 95% confidence interval for the mean assuming that the population standard deviation is not known in advance.

SOLUTION

Following the same three steps discussed earlier, it can be shown that

$$1-\alpha = 0.95 \quad \text{or} \quad \alpha = 1-0.95 = 0.05 \quad \alpha/2 = 0.05/2 = 0.025$$
$$\text{and} \quad 1-\alpha/2 = 1-0.025 = 0.975$$
$$f = n-1 = 41-1 = 40.$$

Thus, from Appendix 5 for Student's t-distribution, it can be observed that

$$t_{\alpha/2,n-1} = t_{0.025,40} = t_{0.975,40} = 2.021.$$

Thus, using Equation 5.23, we can calculate the 95% confidence interval for the mean of the Young's modulus as

$$<\mu>_{0.95} = \left[29{,}576 - 2.021\frac{1{,}507}{\sqrt{41}};\ 29{,}576 + 2.021\frac{1{,}507}{\sqrt{41}}\right]$$
$$= [29{,}100;\ 30{,}052] \text{ ksi.}$$

In this case, the confidence interval for the mean is slightly larger than when the variance is assumed to be known as in Example 5.5a. This is expected; since the variance is assumed to be unknown, more uncertainty is expected in the interval estimation for the mean than when the variance is assumed to be known. However, when the sample size is small, the confidence interval for the case when the variance is unknown is expected to be much larger than when the variance is known, as shown next.

EXAMPLE 5.9

Consider the Young's modulus problem in Example 5.6a. In this case, assume that only 10 samples are taken and the corresponding mean and standard deviation are 29,576 and 1,507 ksi, respectively. However, in this case, assume that the standard deviation is not known in advance.

SOLUTION

Since the standard deviation is unknown, Student's t-distribution needs to be considered.

In this case, $f = 10 - 1 = 9$, and $(1 - \alpha/2) = 0.975$. Again, using Appendix 5, we can show that

$$t_{\alpha/2,n-1} = t_{0.025,9} = t_{0.975,9} = 2.262.$$

Using Equation 5.23, we can calculate the 95% confidence interval for the mean as

$$< \mu >_{0.95} = \left[29{,}576 - 2.262 \frac{1{,}507}{\sqrt{10}};\ 29{,}576 + 2.262 \frac{1{,}507}{\sqrt{10}} \right] \text{ksi}$$
$$= [28{,}498;\ 30{,}654] \text{ ksi.}$$

For a smaller sample size, but with the same sample mean and standard deviation, the interval width is [28,498; 30,654] ksi, larger than [28,642; 30,510] ksi when the variance is assumed to be known as in Example 5.6a.

5.4.4 Lower and Upper Confidence Limit for the Mean with Unknown Variance

As discussed in the previous section, if the information on the variance of the population is not known in advance or if the sample size is small, the probability distribution of $(\bar{X} - \mu)/(\sigma/\sqrt{n})$ will have Student's t-distribution. Thus, Equations 5.19 and 5.20 need to be modified by replacing k_α with $t_{\alpha, n-1}$.

Thus, the $(1 - \alpha)$ lower confidence limit for the mean with unknown variance can be calculated as

$$\mu >_{1-\alpha} = \bar{x} - t_{\alpha, n-1} \frac{s}{\sqrt{n}}. \tag{5.24}$$

Similarly, the $(1 - \alpha)$ upper confidence limit for the mean with unknown variance is

$$< \mu_{1-\alpha} = \bar{x} + t_{\alpha, n-1} \frac{s}{\sqrt{n}}. \tag{5.25}$$

EXAMPLE 5.10

Suppose that for a sample size of 10, the mean and standard deviation of the Young's modulus are estimated to be 29,576 and 1,507 ksi, respectively. Calculate the 95% lower and upper confidence limits for the mean of the Young's modulus.

SOLUTION

Again, following the three steps discussed earlier and using Appendix 5, we can observe that

$$t_{\alpha, n-1} = t_{0.05,9} = t_{0.95,9} = 1.833$$

and

$$t_{\alpha, n-1} \frac{s}{\sqrt{n}} = 1.833 \frac{1{,}507}{\sqrt{10}} = 874 \text{ ksi.}$$

The $(1 - \alpha)$ lower confidence limit for the mean of the Young's modulus with unknown variance is

$$\mu >_{0.95} = 29{,}576 - 874 = 28{,}702 \text{ ksi.}$$

Similarly, the $(1 - \alpha)$ upper confidence limit for the mean of the Young's modulus with unknown variance is

$$< \mu_{0.95} = 29{,}576 + 874 = 30{,}450 \text{ ksi.}$$

5.4.5 Sample Sizes in Estimating the Confidence Interval of Mean

In the discussion of estimating the confidence interval for the mean, the basic objective was to establish how good the sample mean obtained from a sample of size n is with respect to the unknown population mean μ. Obviously, if the width of the interval is small, the sample mean is a good predictor of the population mean. In practical applications, several questions are of considerable interest: (a) If the confidence interval is too wide for a given confidence level, how many additional samples are necessary to reduce the interval by a certain amount, such as 10%? (b) If the width of the confidence interval is reasonable but the confidence level in the prediction needs to be increased, for example, from 95 to 99%, how many additional samples are necessary? (c) If the confidence interval is selected to be, say, ±1% of the sample mean with a given confidence level, such as 95%, what is the minimum required sample size?

The information required to answer these questions was discussed in previous sections. However, it is necessary to elaborate on the selection of the sample size for each of these cases.

Consider again the example on the Young's modulus. For a sample size of 41, the sample mean and the standard deviation are estimated to be 29,576 and 1,507 ksi, respectively. Also assume that the underlying distribution of the population is normal and the information on the variance is available in advance.

(a) Reducing the Confidence Interval

The 95% confidence interval for the mean, as shown in Example 5.5a, is found to be [29,115; 30,037] ksi. Suppose that for the same confidence level, the interval needs to be reduced by 10%. The new interval = (30,037 − 29,115) × 0.9 = 829.8 ksi. Thus,

$$\frac{\sigma}{\sqrt{n}} k_{\alpha/2} = \frac{1{,}507}{\sqrt{n}} k_{0.025} = \frac{1{,}507}{\sqrt{n}} 1.96 = \frac{829.8}{2}$$

or, $n \approx 51$. Thus, an additional 51 − 41 = 10 samples need to be collected.

(b) Increasing the Confidence Level

Suppose that for the example in Part (a), with the same confidence interval of [29,115; 30,037] ksi, the confidence level needs to be increased to 99%.

The new α value is 0.01, and as shown in Example 5.5b, $k_{0.005} = 2.58$. Thus,

$$\frac{\sigma}{\sqrt{n}} k_{\alpha/2} = \frac{1{,}507}{\sqrt{n}} 2.58 = \frac{(30{,}037 - 29{,}115)}{2}$$

or, $n \approx 72$. Thus, an additional 72 − 41 = 31 samples need to be collected.

(c) For a Given Confidence Interval

Suppose that for the example in Part (a), the confidence interval is ±1% of the calculated sample mean. In this case, the interval for the sample mean is [29,576 − 296; 29,576 + 296] ksi. For a 95% confidence level, the required sample size is

$$\frac{\sigma}{\sqrt{n}} k_{\alpha/2} = \frac{1,507}{\sqrt{n}} k_{0.025} = \frac{1,507}{\sqrt{n}} 1.96 = 296$$

or, $n \approx 100$. Thus, an additional $100 - 41 = 59$ samples need to be collected.

The discussion in this section is also valid if the population variance is unknown, except that $k_{\alpha/2}$ needs to replaced by $t_{\alpha/2,\, n-1}$.

5.4.6 Interval Estimation for the Variance

The accuracy of estimating the population variance from a sample of size n is also of concern; similar to the population mean, the sample variance can be used to evaluate the confidence interval and lower and upper confidence limits for the variance. However, the mathematics of estimating the confidence interval and the lower and upper limits for the variance are expected to be much more involved. If the underlying population is normal, irrespective of whether the sample size is small or large, these quantities can be estimated exactly as discussed next.

Suppose X is a normal random variable and a sample of size n (i.e., $x_1, x_2, ..., x_n$) is collected to estimate the unknown but constant population variance σ^2. As in Section 5.4.1, these samples may be assumed to come from n independent normal random variables (i.e., $X_1, X_2, ..., X_n$). The unbiased sample variance can be calculated as

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \tag{5.26}$$

where $\bar{X}$ is the sample mean and is a normal random variable. Assume that μ is the population mean, which is unknown but constant.

The sample variance S^2 is a random variable, and its distribution needs to be determined when X is normal. Equation 5.26 can be rewritten as

$$\begin{aligned} (n-1)S^2 &= \sum_{i=1}^{n} \left[(X_i - \mu) - (\bar{X} - \mu) \right]^2 \\ &= \sum_{i=1}^{n} (X_i - \mu)^2 - n(\bar{X} - \mu)^2. \end{aligned} \tag{5.27}$$

Dividing both sides of Equation 5.27 by σ^2 results in

$$\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 - \left(\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \right)^2. \tag{5.28}$$

Since the population mean μ and the standard deviation σ are unknown but constant, the first term on the right-hand side of Equation 5.28 is the sum of the squares of n independent standard normal variables and has a chi-square distribution with n degrees of freedom (Ang and Tang, 1975). It is denoted as χ_n^2. The second term on the right-hand side of Equation 5.28 is also the square of a standard normal variable and has a chi-square distribution with one degree of freedom. The sum of two chi-square distributions with p and q degrees of freedom is also a chi-square distribution with $(p + q)$

degrees of freedom (Hoel, 1962). Thus, the left-hand side of Equation 5.28, that is, $[(n-1)S^2/\sigma^2]$, has a chi-square distribution with $f = (n-1)$ degrees of freedom. It is denoted as χ^2_{n-1}.

If a random variable C has a chi-square distribution with f degrees of freedom, its PDF is given by

$$f_C(c) = \frac{1}{2^{f/2}\Gamma(f/2)} c^{(f/2-1)} e^{-c/2}, \quad c \geq 0. \tag{5.29}$$

The PDFs of the chi-square distribution for $f = 2, 4, 10, 20$, and 30 are plotted in Figure 5.9. As expected, by virtue of the central limit theorem, the chi-square distribution approaches the normal distribution as $f \to \infty$. The CDFs and the values of the chi-square distribution for various degrees of freedom f are given in Appendix 3. As discussed earlier, the PDF of $(n-1)\,S^2/\sigma^2$ has a chi-square distribution with $(n-1)$ degrees of freedom, as shown in Figure 5.10. Thus, the two-sided $(1-\alpha)$ confidence interval for the population variance σ^2 can be expressed as

$$P\left[c_{\alpha/2,n-1} \leq \frac{(n-1)S^2}{\sigma^2} \leq c_{1-\alpha/2,n-1}\right] = 1-\alpha. \tag{5.30a}$$

Noting that the observed sample variance s^2 is a realization of S^2, and taking the reciprocal of Equation 5.30a, results in

$$P\left[\frac{(n-1)s^2}{c_{1-\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{c_{\alpha/2,n-1}}\right] = 1-\alpha. \tag{5.30b}$$

The two-sided $(1-\alpha)$ for the variance can be given by

$$<\sigma^2>_{1-\alpha} = \left[\frac{(n-1)s^2}{c_{1-\alpha/2,n-1}}; \quad \frac{(n-1)s^2}{c_{\alpha/2,n-1}}\right]. \tag{5.31}$$

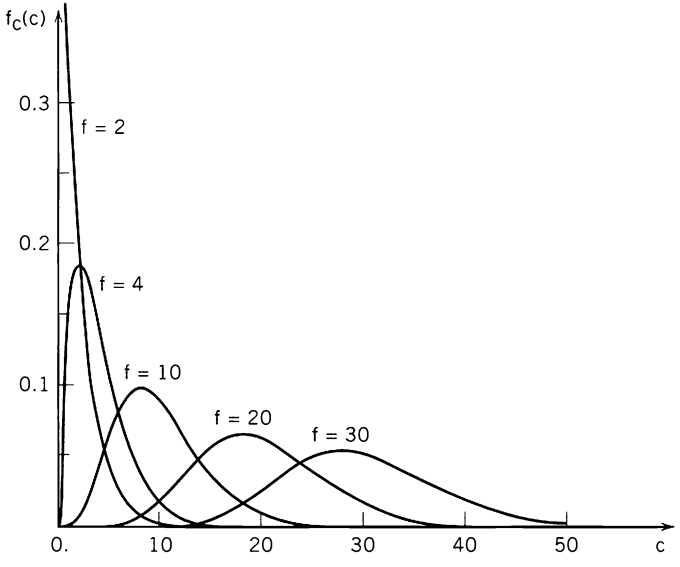


Figure 5.9 PDF of Chi-Square Distribution with f Degrees of Freedom

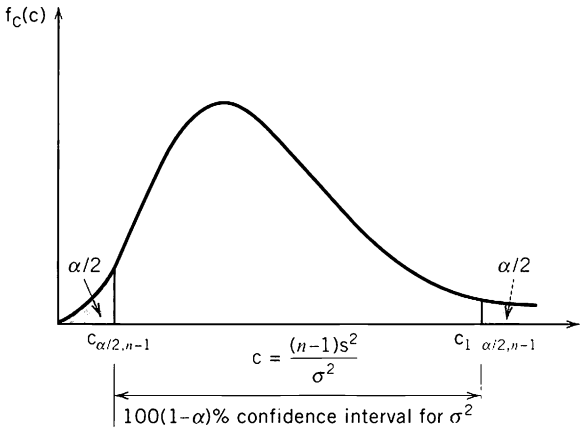


Figure 5.10 Two-sided Confidence Interval for σ^2

Using the concept discussed in Sections 5.4.2 and 5.4.4, the lower and upper confidence limit for the population variance can be calculated similarly. We can show the lower confidence limit for σ^2 to be

$$\sigma^2 > _{1-\alpha} = \frac{(n-1)s^2}{c_{1-\alpha,n-1}}. \tag{5.32}$$

Equation 5.32 states that the population variance σ^2 is at least $\sigma^2>_{1-\alpha}$ with a confidence level of $1 - \alpha$.

Similarly, we can show the upper confidence limit for σ^2 to be

$$< \sigma^2 _{1-\alpha} = \frac{(n-1)s^2}{c_{\alpha,n-1}}. \tag{5.33}$$

Equation 5.33 states that the population variance σ^2 is less than $<\sigma^2_{1-\alpha}$ with a confidence level of $1 - \alpha$.

EXAMPLE 5.11

Assume that the Young's modulus considered in the previous sections has a normal distribution. Using 41 sample points, we find its variance to be $1{,}507^2$ ksi^2. Determine (a) the 95% confidence interval for the population variance σ^2, (b) the 95% lower confidence limit for σ^2, and (c) the 95% upper confidence limit for σ^2.

SOLUTION

(a) $f = 41 - 1 = 40$, and $1 - \alpha = 0.95$. Thus, $\alpha = 0.05$, and $\alpha/2 = 0.025$. According to Appendix 3, $c_{0.025,\,40} = 24.4$, and $c_{0.975,\,40} = 59.3$. Using Equation 5.31, we find the 95% confidence interval for the population variance to be

$$< \sigma^2 >_{0.95} = \left[\frac{(41-1)1{,}507^2}{59.3};\ \frac{(41-1)1{,}507^2}{24.4} \right]$$

$$= [1{,}531{,}905;\ 3{,}723{,}031]\ \text{ksi}^2.$$

(b) Again, $f = 41 - 1 = 40$, and $\alpha = 0.05$. From Appendix 3, $c_{0.95,40} = 55.8$. Using Equation 5.32, we find the 95% lower confidence limit for the population variance to be

$$\sigma^2 > _{0.95} = \frac{(41-1)1,507^2}{55.8} = 1,627,992 \text{ ksi}^2.$$

(c) In this case, from Appendix 3, $c_{0.05,40} = 26.5$. Using Equation 5.33, we find the 95% upper confidence limit for the population variance to be

$$< \sigma^2_{0.95} = \frac{(41-1)1,507^2}{26.5} = 3,427,998 \text{ ksi}^2.$$

5.5 CONCLUDING REMARKS

Modeling and quantifying uncertainties in random variables are the initial and essential steps in any risk-based analysis and design. Collecting data and extracting information from the data in terms of many descriptors are introduced in this chapter.

Descriptions of randomness in terms of many numerical descriptors are routine in the profession. Procedures to select the underlying distribution of the random variables, particularly when several of the standard distributions are likely choices, are introduced in this chapter. To describe a particular distribution, its parameters need to be determined; procedures to do this are also discussed.

The two most commonly used methods of point estimation of parameters, that is, the method of moments and the method of maximum likelihood, are introduced in this chapter. The accuracy of estimating the mean and variance of a population using a sample size of n is discussed. Procedures to calculate the confidence interval and the lower and upper confidence limits are also presented. The sample size required for a prediction with a preselected confidence level is explained.

The uncertainties in most of the random variables used in engineering practice, in terms of their distribution and parameters, have already been quantified by many researchers. For example, the Young's modulus of all grades of steel can be considered to have a lognormal distribution with a mean value of 29,000 ksi and a coefficient of variation of 0.06. In other cases, the underlying distribution of a random variable may have been established, but its parameters will change from location to location. For example, as discussed in Chapter 4, the maximum annual wind velocity can be considered to have a Type I extreme value distribution. However, its parameters will depend on location and need to be estimated using the wind velocity data available at a particular location. In other cases dealing with parameters whose uncertainty has not been studied before, both the distribution and its parameters need to be evaluated using the procedures discussed in this chapter.

5.6 PROBLEMS

5.1 The test scores of a class consisting of 30 students are given in Problem 3.1.

(a) Plot the data on normal probability paper. Is the normal distribution acceptable in this case? Estimate the mean and standard deviation from the plot.

(b) Plot the data on lognormal probability paper. Is the lognormal distribution acceptable?

5.2 The annual rainfall in a city is assumed to be uniformly likely between 100 and 150 cm. The data on annual rainfall for the city for the last 20 years are as follows: 102, 110, 117, 133, 105, 144, 149, 145, 127, 124, 122, 141, 138, 119, 119, 106, 114, 130, 106, and 136.

(a) What kind of probability paper do you need?

(b) Plot the data on the probability paper. Is the uniform distribution acceptable based on this plotting?

(c) Perform a chi-square test on the data. Is the uniform distribution acceptable at a 1% significance level?

5.3 The time between replacement, T, in days for light bulbs in a particular building is under consideration. Suppose T is assumed to have an exponential distribution given by its PDF as

$$f_T(t) = \nu\, e^{-\nu t},\ t \geq 0.$$

(a) Construct an exponential probability paper.

(b) Suppose the following data on T are available: 72, 50, 320, 100, 25, 450, 190, 245, 140, 300, 10, 900, 275, 200, 30, 620, 380, 150, 120, 85, and 55.

Plot the data on the probability paper. Is an exponential distribution acceptable for T?

5.4 Perform the chi-square test on the data given in Problem 5.3.

(a) Is an exponential distribution acceptable at a 5% significance level?

(b) Is an exponential distribution acceptable at a 1% significance level?

5.5 Perform the chi-square test on the data given in Problem 3.1.

(a) Can the underlying distribution of the test score be accepted as normal at a 5% significance level?

(b) Can the underlying distribution of the test score be accepted as lognormal at a 5% significance level?

5.6 The annual precipitation during the past 30 years in Tucson, Arizona, is given in Problem 3.2.

(a) Perform the chi-square test on the data, assuming a normal distribution.

(b) Can the underlying distribution be accepted as normal at a 5% significance level?

(c) Is the normal distribution acceptable at a 1% significance level?

(d) Can the underlying distribution be accepted as lognormal at a 5% significance level?

5.7 The total numbers of accidents per month at an intersection for a 2-year period are 0, 2, 0, 0, 0, 0, 0, 1, 1, 1, 1, 2, 1, 1, 1, 1, 3, 1, 1, 1, 1, 1, 4, and 0.

(a) Perform the chi-square test on the data, assuming a Poisson distribution.

(b) Is a Poisson distribution acceptable at a 5% significance level?

5.8 Perform the K–S test on the data given in Problem 3.1 for normal distribution. Is normal distribution acceptable at a 5% significance level?

5.9 Perform the K–S test on the data given in Problem 3.1 for lognormal distribution. Is lognormal distribution acceptable at a 5% significance level?

5.10 Assume the test scores given in Problem 3.1 can be modeled by a normal random variable. Estimate its two parameters using the method of moments and using the method of maximum likelihood.

5.11 Assume the test scores given in Problem 3.1 can be modeled by a lognormal random variable. Estimate its two parameters using the method of moments and using the method of maximum likelihood.

5.12 The time intervals, T, in days between two successive accidents in an intersection were found to be 5, 10, 1, 6, 3, 7, and 5. Suppose T can be modeled by the geometric distribution. Determine its parameter using the method of moments, and using the method of maximum likelihood.

5.13 In Problem 5.7, calculate the parameter of the Poisson random variable using the method of moments and using the method of maximum likelihood.

5.14 The annual precipitation during the past 30 years in Tucson, Arizona, is given in Problem 3.2.

(a) Plot the data on normal probability paper. Is the normal distribution acceptable?

(b) Calculate the mean and standard deviation of the annual precipitation from the plot.

(c) Determine the 99% confidence interval for the mean of the annual precipitation.

5.15 The posted speed limit on a particular segment of an interstate highway is 65 mph. From past experience, the standard deviation of vehicle speed is 10 mph. To estimate the average speed of vehicles, the speeds of 50 vehicles are recorded and the average speed is estimated to be 67 mph.

(a) Determine the 95% confidence interval for the mean vehicle speed.

(b) Determine the 95% lower and upper confidence limits for the mean vehicle speed.

(c) The confidence interval estimated in Part (a) needs to be reduced by 20%. How many additional measurements are needed?

(d) The confidence level of the confidence interval estimated in Part (a) needs to be increased from 95% to 99%. How many additional measurements are required?

5.16 Vehicle speed needs to be estimated with an accuracy of ±5 mph of the average speed with a 99% confidence level. Assume the standard deviation of the vehicle speed is 10 mph. How many samples are required?

5.17 The daily water consumption in a desert city is studied for 30 days. The mean daily water consumption is found to be 10 million gallons per day (mgd), and the corresponding standard deviation is 2 mgd. No prior information on the standard deviation of daily water consumption of the city is available.

(a) Determine the 95% confidence interval for the mean daily water consumption.

(b) Determine the 95% lower and upper confidence limit for the mean daily water consumption.

(c) The confidence interval of the mean obtained in Part (a) needs to be reduced by 10%. How many additional samples are required?

5.18 The biological oxygen demand (BOD) level is measured for 10 days for a river at a station and is found to be 3.6, 4.2, 2.8, 4.5, 3.0, 2.9, 2.8, 5.0, 3.1, and 3.3 mg/L. Assume that the daily BOD level is a normal random variable.

(a) Calculate the mean and standard deviation of the daily BOD level.

(b) Determine the 99% confidence interval for the mean BOD.

(c) Determine the 99% upper confidence limit for the mean BOD.

(d) Determine the 99% confidence interval for the variance.

5.19 Load tests are conducted on 15 piles at a large construction project. The capacity of each pile is 80, 90, 75, 93, 100, 76, 89, 80, 82, 87, 91, 98, 84, 86, and 81 kip. No prior information on the variance of the pile capacity is available. Assume the pile capacity can be modeled as a normal variable.

(a) Calculate the mean and variance of the pile capacity.

(b) Calculate the 95% confidence interval for the mean.

(c) Calculate the 95% lower confidence limit for the mean.

(d) Calculate the 95% confidence interval for the variance.

(e) Calculate the 95% lower confidence limit for the variance.

5.20 How will Problem 5.19 change if the variance of the pile capacity, known in advance, is exactly equal to the value obtained from the data? Recalculate Parts (b) and (c) of Problem 5.19.

Chapter 6

Randomness in Response Variables

6.1 INTRODUCTORY COMMENTS

Modeling the uncertainty in random variables based on the available data was discussed in detail in Chapter 5. However, in many engineering problems, the uncertainty associated with one random variable needs to be estimated indirectly from the information on uncertainty in another random variable. For example, wind velocity in miles per hour (mph) is measured continuously at airports. To design a structure for wind load, an engineer needs to calculate the wind pressure on the structure, usually in pounds per square foot (psf); however, no direct information is available on the statistics of wind pressure. By collecting data on wind velocity and following the procedures discussed in Chapter 5, the engineer can easily quantify the uncertainty in the wind velocity in terms of its mean; its variance, standard deviation, or coefficient of variation; and the underlying distribution. However, for probabilistic design, the uncertainty in the wind pressure needs to be estimated from the uncertainty in the wind velocity. This type of analysis is discussed in Chapter 6.

According to the American Society of Civil Engineers (ASCE) standard ASCE 7-95 (Minimum Design Loads for Buildings and Other Structures), the design wind pressure at height z can be calculated as

$$q_z = 0.00256 K_z K_{zt} V^2 I \tag{6.1}$$

where q_z is the wind pressure in psf, K_z is the velocity pressure exposure coefficient at height z, K_{zt} is the topographic factor, V is the basic wind speed in mph, and I is the importance factor. Initially, assuming K_z, K_{zt}, and I are constant, q_z is a function of one random variable V, whose uncertainty is completely defined by its known PDF. In this case, the nonlinear relationship between q_z and V is known and is given by Equation 6.1. Since V is a random variable, q_z must be a random variable.

In this example, wind pressure can be considered to be the response variable, and it is functionally related to only one basic random variable, or simply one random vari-

able (i.e., wind velocity). In general, the response variable could be functionally related to more than one random variable. This will be the case when K_z, K_{zt}, I, and V in Equation 6.1 are assumed to be random variables. In most engineering problems, functional relationships (linear or nonlinear) between the response and basic random variables are known; however, in some cases, the exact relationship may not be known explicitly. Since the response variable is a function of other random variables, it will also be random, whether the exact functional relationship between them is known or not. The subject of this chapter is the quantification of the uncertainty in the response variable when it is related to other random variables with a known or unknown relationship.

6.2 KNOWN FUNCTIONAL RELATIONSHIP BETWEEN THE RESPONSE AND A SINGLE BASIC RANDOM VARIABLE

6.2.1 Linear Relationship

Suppose the response variable Y has a known linear relationship with a single random variable X given by

$$Y = a + bX, \tag{6.2}$$

where a and b are known constants. Since X is a random variable, Y will also be a random variable. For the linear functional relationship, it can be shown that Y will have the same distribution as X. The mean or the expected value of Y, $E(Y)$, can be calculated as

$$E(Y) = \int_{-\infty}^{+\infty} (a + bX) f_X(x)dx = a \int_{-\infty}^{+\infty} f_X(x)dx + b \int_{-\infty}^{+\infty} x f_X(x)dx.$$

Since area under the PDF is 1.0 and the first moment of the PDF about the origin is the mean or expected value, the expected value of Y can be calculated as

$$E(Y) = a + bE(X). \tag{6.3}$$

The variance of Y can be shown to be

$$\begin{aligned} \mathrm{Var}(Y) &= \int_{-\infty}^{+\infty} \{(a + bx) - [a + bE(X)]\}^2 f_X(x)dx \\ &= b^2 \int_{-\infty}^{+\infty} [x - E(X)]^2 f_X(x)dx. \end{aligned}$$

Since the integral in the preceding equation is the variance of X, the variance of Y in Equation 6.2 becomes

$$\mathrm{Var}(Y) = b^2 \mathrm{Var}(X). \tag{6.4}$$

If X is a normal random variable, the response variable Y is also a normal random variable with mean and variance obtained using Equations 6.3 and 6.4, respectively.

EXAMPLE 6.1

Shallow strip footing is frequently used for ordinary buildings. Suppose a strip footing of width B and depth H ($H \leq B$) from the ground surface needs to be located in a dense

sand layer. Considering general shear failure, we can calculate the ultimate bearing capacity of the soil, q_u, using Terzaghi's (1943) bearing capacity equation as

$$q_u = cN_c + \gamma HN_q + \frac{1}{2}\gamma BN_\gamma$$

where c is the cohesion, γ is the unit weight of the soil, and N_c, N_q, and N_γ are the bearing capacity factors and can be estimated from the information on the angle of internal friction ϕ of the soil. Suppose the soil layer has a ϕ of 20° and the corresponding bearing capacity factors are 17.69, 7.44, and 3.64, respectively. Further assume that $\gamma = 115$ pcf, $H = 3$ feet, and $B = 4$ feet. All the parameters in Terzaghi's bearing capacity equation are considered to be constant except c; c is assumed to have a normal distribution with a mean and COV of 400 psf and 0.1, respectively.

(a) Determine the distribution of q_u and its mean and standard deviation.

(b) A safety factor of 3 is generally used to calculate the allowable bearing capacity, q_a (i.e., $q_a = q_u/3$). What is the distribution of q_a and its mean and standard deviation?

(c) What is the probability that the allowable bearing capacity is less than 3,000 psf?

SOLUTION

(a) $q_u = c \times 17.69 + 115 \times 3 \times 7.44$

$$+\frac{1}{2} \times 115 \times 4 \times 3.64 = 3,404 + 17.69c$$

Thus,

$$E(q_u) = 3,404 + 17.69 \times 400 = 10,480 \text{ psf}$$

And

$$\text{Var}(q_u) = 17.69^2 \times \text{Var}(c) = 17.69^2 \times (400 \times 0.1)^2 = 500697.76$$

or

$$\sigma_{q_u} = 17.69 \times 40 = 707.6 \text{ psf}.$$

Thus, q_u is normal with a mean of 10,480 psf and a standard deviation of 707.6 psf.

(b) $q_a = \frac{q_u}{3}$. Thus, q_a is also a normal random variable with a mean of 10,480/3 = 3,493.3 psf and a standard deviation of 707.6/3 = 235.9 psf.

(c) $P(q_a \le 3,000) = \Phi\left(\frac{3,000 - 3,493.3}{235.9}\right) = \Phi(-2.09) = 1 - 0.98169 = 0.01831.$

6.2.2 Nonlinear Relationship

A more general case, in which the functional relationship between the response variable and the basic random variable is not linear, is considered next. To generalize the discussion, assume that the response variable Y is functionally related to X as

$$Y = g(X). \tag{6.5}$$

If Y is a monotonically increasing function of X, then

$$P(Y \le y) = P(X \le x)$$

or

$$F_Y(y) = F_X(x) = F_X[g^{-1}(y)].$$

The value $g^{-1}(y)$ can be evaluated by inverting Equation 6.5. If both sides are differentiated with respect to y, the PDF of Y can be obtained as

$$f_Y(y) = f_X[g^{-1}(y)]\frac{dg^{-1}(y)}{dy}. \tag{6.6}$$

Thus, if the functional relationship g and the PDF of X are known, the uncertainty in Y in terms of its PDF can be obtained from Equation 6.6.

If Y decreases with X, $dg^{-1}(y)/dy$ can be negative. Since the PDF of a random variable cannot be negative, its absolute value is of interest. Therefore, to account for both cases, the PDF of Y is written as

$$f_Y(y) = f_X[g^{-1}(y)]\left|\frac{dg^{-1}(y)}{dy}\right|. \tag{6.7}$$

In many cases, the inverse function $g^{-1}(y)$ may have n values x_i, and if the $f_X(x_i)$ are nonzero positive numbers, Equation 6.7 needs to be modified as

$$f_Y(y) = \sum_{i=1}^{n} f_X[g_i^{-1}(y)]\left|\frac{dg_i^{-1}(y)}{dy}\right|. \tag{6.8}$$

EXAMPLE 6.2

Similar to Equation 6.1, assume the following quadratic relationship between Y and X:

$$Y = cX^2 \tag{6.9}$$

where c is a constant. Initially, assume X is a lognormal random variable with parameters λ_X and ζ_X, and its PDF is given by

$$f_X(x) = \frac{1}{\zeta_X\sqrt{2\pi}}\frac{1}{x}\exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda_X}{\zeta_X}\right)^2\right], \quad x \geq 0.$$

If Equation 6.9 is inverted, the two roots of X can be shown to be

$$x = g^{-1}(y) = \pm\left(\sqrt{\frac{y}{c}}\right)$$

and

$$\frac{dx}{dy} = \frac{dg^{-1}(y)}{dy} = \pm\frac{1}{2\sqrt{cy}}.$$

It is discussed in Chapter 4 that the lognormal distribution will have a nonzero PDF only for positive values of X. Thus, considering only the positive root of X and the absolute value of dx/dy and using Equation 6.7, we can show the PDF of Y to be

$$f_Y(y) = f_X\left(\sqrt{\frac{y}{c}}\right)\frac{1}{2\sqrt{cy}}$$

or

$$f_Y(y) = \frac{1}{\zeta_X\sqrt{2\pi}}\frac{1}{\sqrt{\frac{y}{c}}}\exp\left[-\frac{1}{2}\left(\frac{\ln\left(\sqrt{\frac{y}{c}}\right) - \lambda_X}{\zeta_X}\right)^2\right]\frac{1}{2\sqrt{cy}}, \quad y \geq 0. \tag{6.10}$$

On the other hand, if X is a normal random variable with mean μ_X and standard deviation σ_X, its PDF can be shown to be

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right], \quad -\infty < x <$$

For a normal distribution, the PDF exists at both the positive and negative roots, and the PDF of Y becomes

$$f_Y(y) = \left[f_X\left(+\sqrt{\frac{y}{c}}\right) + f_X\left(-\sqrt{\frac{y}{c}}\right)\right]\frac{1}{2\sqrt{cy}}$$

or

$$f_Y(y) = \frac{1}{\sigma_X\sqrt{2\pi}}\frac{1}{2\sqrt{cy}}\left\{\exp\left[-\frac{1}{2}\left(\frac{+\sqrt{\frac{y}{c}}-\mu_X}{\sigma_X}\right)^2\right]\right.$$

$$\left.+\exp\left[-\frac{1}{2}\left(\frac{-\sqrt{\frac{y}{c}}-\mu_X}{\sigma_X}\right)^2\right]\right\}, \quad -\infty < y < +\infty. \tag{6.11}$$

Although the PDFs of the response variable Y for lognormal and normal random variables appear to be complicated, they are not difficult to obtain, and no approximation is involved in their evaluation.

6.3 RESPONSE AS A KNOWN FUNCTION OF MULTIPLE RANDOM VARIABLES

When the response variable is a function of multiple random variables, its uncertainty analysis is quite involved. A closed-form analytical solution can be obtained only in special cases. Some of the cases in which closed-form solutions can be obtained are (a) multiple random variables with known joint PDF and explicit functional relationships between the response and basic random variables, (b) sums and differences of independent normal random variables, (c) products and quotients of independent lognormal random variables, and (d) sum of independent Poisson random variables. These cases are discussed briefly in Section 6.3.1. In many other cases, only partial statistical information of the response variable can be obtained in terms of its mean and variance. Sometimes the mean and variance can be only be estimated approximately, as discussed in Section 6.4.

6.3.1 Exact Solution

6.3.1.1 Known Joint Density Function and Functional Relationship

Assume two random variables X_1 and X_2 have a known joint PDF of $f_{X_1X_2}(x_1, x_2)$. They are functionally related to two response random variables Y_1 and Y_2 as

$$y_1 = y_1(x_1, x_2)$$
$$y_2 = y_2(x_1, x_2).$$

Further assume, as in Section 6.2.2, that unique inverses of these functions exist:

$$x_1 = x_1(y_1, y_2)$$
$$x_2 = x_2(y_1, y_2).$$

Then, it can be shown that

$$f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}(x_1, x_2)\,|J| \tag{6.12}$$

where $|J|$ is the determinant of the Jacobian J defined as

$$|J| = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_2}{\partial y_1} \\ \dfrac{\partial x_1}{\partial y_2} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} = \frac{\partial x_1}{\partial y_1} \cdot \frac{\partial x_2}{\partial y_2} - \frac{\partial x_2}{\partial y_1} \cdot \frac{\partial x_1}{\partial y_2}. \tag{6.13}$$

Equation 6.12 is rarely used since the joint PDF and the functional relationships between the random variables are seldom known or easily invertible.

6.3.1.2 Sums and Differences of Independent Normal Variables

Suppose that X_1 and X_2 are statistically independent normal random variables with means μ_X and μ_{X_2}, respectively, and corresponding standard deviations σ_{X_1} and σ_{X_2}. They are functionally related to the response variable Y as

$$Y = g(X_1, X_2) = X_1 + X_2. \tag{6.14}$$

When X_i's are statistically independent, the CDF of Y can be calculated as

$$F_Y(y) = \iint_{x_1 + x_2 \le y} f_{X_1}(x_1) f_{X_2}(x_2)\, dx_1 dx_2. \tag{6.15}$$

If we change the variable of integration from x_1 to y, Equation 6.15 becomes

$$F_Y(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_{X_1}[g^{-1}(y, x_2)] f_{X_2}(x_2) \left| \frac{\partial g^{-1}(y, x_2)}{\partial y} \right| dy\, dx_2. \tag{6.16}$$

The corresponding PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}[g^{-1}(y, x_2)] f_{X_2}(x_2) \left| \frac{\partial g^{-1}(y, x_2)}{\partial y} \right| dx_2. \tag{6.17}$$

Equation 6.14 can be used to show that

$$g^{-1}(y, x_2) = x_1 = y - x_2$$

and

$$\frac{\partial g^{-1}(y, x_2)}{\partial y} = \frac{\partial x_1}{\partial y} = 1.$$

Thus, Equation 6.17 becomes

$$f_Y(y) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2} \left\{ \left(\frac{y - x_2 - \mu_{X_1}}{\sigma_{X_1}} \right)^2 + \left(\frac{x_2 - \mu_{X_2}}{\sigma_{X_2}} \right)^2 \right\} \right] dx_2. \tag{6.18}$$

After simplification, Equation 6.18 becomes

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}} \exp\left[-\frac{1}{2} \left\{ \frac{y - (\mu_{X_1} + \mu_{X_2})}{\sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}} \right\}^2 \right]. \tag{6.19}$$

From Equation 6.19, it is clear that Y is also a normal random variable with a mean of

$$\mu_Y = \mu_{X_1} + \mu_{X_2}$$

and a corresponding variance of

$$\sigma_Y^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2.$$

The preceding observations can be generalized. Suppose that the X_i's are statistically independent normal random variables with mean μ_{X_i} and standard deviation σ_{X_i}, and they are functionally related to the response variable Y as

$$Y = a_1X_1 + a_2X_2 + \ldots + a_nX_n \tag{6.20}$$

where a_i's are constants. It can be shown that Y is also a normal random variable with mean and variance of

$$\mu_Y = \sum_{i=1}^{n} a_i \mu_{X_i} \tag{6.21}$$

and

$$\sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2. \tag{6.22}$$

EXAMPLE 6.3

Assume a random variable Y can be represented by the following relationship:

$$Y = X_1 + 2X_2 - 4X_3$$

where X_1, X_2, and X_3 are statistically independent normal variables with mean values of 1.0, 1.5, and 0.8, respectively, and corresponding standard deviations of 0.10, 0.20, and 0.15, respectively. Then Y is also a normal random variable, and its mean and standard deviation are

$$\mu_Y = 1.0 + 2 \times 1.5 - 4 \times 0.8 = 0.8$$

and

$$\sigma_Y = \sqrt{0.1^2 + 2^2 \times 0.2^2 + 4^2 \times 0.15^2} = 0.728.$$

With this information, any probability of Y can be calculated, as discussed in Section 4.2.1.

EXAMPLE 6.4

Consider a weight that is hung by a cable. The load-carrying capacity or the resistance of the cable, R, is a normal random variable with a mean and standard deviation of 120 kip and 18 kip, respectively. The load effect, S, is also a normal random variable with

a mean and standard deviation of 50 kip and 12 kip, respectively. Assume that R and S are statistically independent. The cable will break when its resistance is smaller than the applied load. Another variable Y can be introduced as

$$Y = R - S.$$

As discussed here, since R and S are independent normal random variables, Y will also be a normal random variable with $\mu_Y = 120 - 50 = 70$ kip, and $\sigma_Y = \sqrt{18^2 + 12^2} = 21.63$ kip. Thus, the cable will break when $Y \leq 0$, and its probability can be calculated as

$$P(Y \leq 0) = \Phi\left(\frac{0 - 70}{21.63}\right) = \Phi(-3.24) = 1 - 0.99940 = 0.00060.$$

6.3.1.3 Products and Quotients of Independent Lognormal Variables

In this case, assume that

$$Y = X_1 X_2 \ldots X_n = \prod_{i=1}^{n} X_i \tag{6.23}$$

where X_i's are statistically independent lognormal variables with parameters λ_{X_i} and ζ_{X_i} (as discussed in Section 4.2.2, these parameters can be estimated from the information on the corresponding mean and standard deviation). Following procedures similar to Section 6.3.1.2 for sums and differences of independent normal variables, we can show that the response variable Y is also a lognormal random variable with the following parameters:

$$\lambda_Y = \sum_{i=1}^{n} \lambda_{X_i} \tag{6.24}$$

$$\zeta_Y^2 = \sum_{i=1}^{n} \zeta_{X_i}^2. \tag{6.25}$$

EXAMPLE 6.5

Assume that the random variable Y can be represented by the following relationship:

$$Y = \frac{X_1 \bullet X_2}{X_3}$$

where X_1, X_2, and X_3 are statistically independent lognormal variables with means of 1.0, 1.5, and 0.8, respectively, and corresponding standard deviations of 0.10, 0.20, and 0.15, respectively. Then, using Equations 4.10 and 4.11, the two parameters of each of the three lognormal variables are

$$\zeta_{X_1}^2 = \ln\left(1 + \frac{0.10^2}{1.0^2}\right) = 0.0100$$

$$\zeta_{X_2}^2 = \ln\left(1 + \frac{0.20^2}{1.5^2}\right) = 0.0176$$

$$\zeta_{X_3}^2 = \ln\left(1 + \frac{0.15^2}{0.8^2}\right) = 0.0346$$

and

$$\lambda_{X_1} = \ln(1.0) - \frac{1}{2} \times 0.01 = -0.005$$

$$\lambda_{X_2} = \ln(1.5) - \frac{1}{2} \times 0.0176 = 0.3967$$

$$\lambda_{X_3} = \ln(0.8) - \frac{1}{2} \times 0.0346 = -0.2404.$$

The discussion in this section suggests that Y is a lognormal random variable, and its two parameters can be calculated using Equations 6.24 and 6.25. For the problem under consideration, these parameters are

$$\lambda_Y = \lambda_{X_1} + \lambda_{X_2} - \lambda_{X_3} = -0.005 + 0.3967 - (-0.2404) = 0.6321$$

and

$$\zeta_Y^2 = 0.01 + 0.0176 + 0.0346 = 0.0622.$$

Then,

$$\zeta_Y = 0.249.$$

With this information, any probability of Y can be estimated.

EXAMPLE 6.6

Example 6.4 on the probability of cable failure can be considered again. However, R and S are now assumed to be statistically independent lognormal random variables with the same means and standard deviations as before. The COVs of R and S are 18/120 = 0.15, and 12/50 = 0.24, respectively. Since they are relatively small, less than 0.30, we can assume that $\zeta \approx$ COV, as discussed in Chapter 4. For this example

$$\zeta_R^2 \approx 0.15^2 = 0.0225$$

$$\zeta_S^2 \approx 0.24^2 = 0.0576$$

and

$$\lambda_R = \ln 120 - \frac{1}{2} \times 0.0225 = 4.7762$$

$$\lambda_S = \ln 50 - \frac{1}{2} \times 0.0576 = 3.8832.$$

In this case, another random variable Y can be defined as

$$Y = \frac{R}{S}.$$

Thus, Y is also a lognormal random variable with the following parameters:

$$\lambda_Y = \lambda_R - \lambda_S = 4.7762 - 3.8832 = 0.893$$

$$\zeta_Y = \sqrt{\zeta_R^2 + \zeta_S^2} = \sqrt{0.0225 + 0.0576} = 0.2830.$$

In this case, the cable will break if $Y \leq 1$, and the corresponding probability of failure can be calculated as

$$P(Y \le 1) = \Phi\left(\frac{\ln 1 - 0.893}{0.283}\right) = \Phi(-3.16) = 1 - 0.99921 = 0.00079.$$

This probability of failure is slightly different than when both R and S were normal random variables.

EXAMPLE 6.7

The Reynolds number is used to determine the nature of a flow, that is, whether it is laminar or turbulent. It can be calculated as

$$R = \frac{VD\rho}{\mu}$$

where V is the average velocity in a tube, D is the diameter of the tube, ρ is the mass density of the fluid, and μ is the viscosity of the fluid. Suppose V, D, ρ, and μ are statistically independent lognormal random variables with means of 5 ft/s, 1 foot, 1.94 slug/ft^3, and 10^{-5}slug/sec-ft, respectively, and the corresponding COVs are 0.2, 0.05, 0.05, and 0.1, respectively.

(a) Determine the distribution of R.

(b) What is the probability that R is greater than 10^6?

SOLUTION

(a) As discussed in Section 4.2.2, λ_i and ζ_i are the two parameters of the ith lognormal random variable, and if its COV δ_i is relatively small, $\zeta_i \approx \delta_i$. For the problem under consideration, these parameters for all the lognormal variables can be calculated as

$$\zeta_V \approx 0.2, \quad \zeta_D \approx 0.05, \quad \zeta_\rho \approx 0.05, \quad \zeta_\mu \approx 0.1$$

$$\lambda_V = \ln 5 - 0.5 \times 0.2^2 = 1.58944$$

$$\lambda_D = \ln 1 - 0.5 \times 0.05^2 = -0.00125$$

$$\lambda_\rho = \ln 1.94 - 0.5 \times 0.05^2 = 0.66144$$

$$\lambda_\mu = \ln 10^{-5} - 0.5 \times 0.1^2 = -11.51793.$$

R is also a lognormal random variable, and with Equations 6.24 and 6.25, its two parameters can be calculated as

$$\lambda_R = 1.58944 - 0.00125 + 0.66144 - (-11.51793) = 13.76756$$

and

$$\zeta_R^2 = 0.2^2 + 0.05^2 + 0.05^2 + 0.1^2 = 0.055.$$

Thus, $\zeta_R = 0.23452$.

(b) $$P(R > 10^6) = 1 - P(R \le 10^6) = 1 - \Phi\left(\frac{\ln 10^6 - 13.76756}{0.23452}\right)$$

$$= 1 - \Phi(0.20) = 1 - 0.57926 = 0.42074.$$

6.3.1.4 Sum of Independent Poisson Random Variables

Suppose X_1 and X_2 in Equation 6.14 are statistically independent Poisson random variables with parameters ν_{X_1} and ν_{X_2}, respectively. For the discrete random variable case, similar to Section 6.3.1.2, the PMF of Y can be shown to be

$$\begin{aligned} p_Y(y) &= \sum_{\text{all } x_2} p_{X_1}(y-x_2)p_{X_2}(x_2) \\ &= \sum_{\text{all } x_2} \frac{e^{(-\nu_{X_1}t)}(\nu_{X_1}t)^{y-x_2}}{(y-x_2)!}\frac{e^{(-\nu_{X_2}t)}(\nu_{X_2}t)^{x_2}}{x_2!} \\ &= \frac{e^{-(\nu_{X_1}+\nu_{X_2})t}}{y!}\left[\sum_{\text{all } x_2}\frac{y!}{x_2!(y-x_2)!}(\nu_{X_1}t)^{(y-x_2)}(\nu_{X_2}t)^{x_2}\right]. \end{aligned}$$

The terms within the square brackets in the preceding equation represent a binomial series and can be shown to be equal to $(\nu_{X_1}t+\nu_{X_2}t)^y$. Thus, the PMF of Y becomes

$$p_Y(y) = \frac{\left[\left(\nu_{X_1}+\nu_{X_2}\right)t\right]^y}{y!}e^{-(\nu_{X_1}+\nu_{X_2})t} \tag{6.26}$$

Equation 6.26 shows that Y is also a Poisson random variable with parameter $(\nu_{X_1}+\nu_{X_2})$.

Generalizing the preceding discussion, we can show that if the X_i's are statistically independent Poisson random variables with parameter ν_{X_i}'s, and they are related to the response variable Y as

$$Y = X_1 + X_2 + \ldots + X_n, \tag{6.27}$$

then Y is also a Poisson random variable with parameter

$$\nu_Y = \sum_{i=1}^{n}\nu_{X_i}. \tag{6.28}$$

Note, however, that if Y is the difference of Poisson random variables, then Y is not a Poisson random variable.

EXAMPLE 6.8

A parking garage has three entrances, A, B, and C. The entrance of cars through each gate can be modeled by three independent Poisson random variables. Past records indicate that an average of 5, 10, and 15 cars per hour enter through Entrances A, B, and C, respectively. What is the probability that 40 cars will enter the garage in the next 2 hours?

SOLUTION

Let T be the event representing the total number of cars entering the garage. Then,

$$T = A + B + C.$$

Since T is the summation of three independent Poisson random variables, it is also a Poisson random variable with parameter $\nu_T = 5 + 10 + 15 = 30$ cars per hour. Thus,

$$P(T = 40) = \frac{e^{-(30\times 2)}(30\times 2)^{40}}{40!} = 0.00143.$$

6.3.2 Central Limit Theorem

The discussion of the distribution of the sum and product of random variables would be incomplete without introducing the *central limit theorem*. This important theorem states that the sum of a large number of random variables, where none of them dominates the sum and regardless of their initial distributions, tends to the normal distribution as the number increases. If

$$X = \sum_{i=1}^{n} X_i \tag{6.29}$$

the random variable X tends to have a normal distribution as n approaches ∞.

The concept can be extended to the product of a large number of random variables: where none of them dominates the product and regardless of their initial distributions, the distribution tends to the lognormal distribution as the number increases. If

$$X = \prod_{i=1}^{n} X_i \tag{6.30}$$

the random variable X tends to have a lognormal distribution as n approaches ∞.

6.4 PARTIAL AND APPROXIMATE SOLUTIONS

Besides the cases discussed in Section 6.3, there are many practical problems in which a random variable Y is a function of many other random variables X_i. The functional relationship could be linear or nonlinear; the probabilistic characteristics of Y may not be defined precisely; it may only be possible to obtain limited information. If the distributions of the X_i's are not known, or if X_1 is normal, X_2 is lognormal, and so on, it is not possible to determine the exact distribution of the response variable Y as in Equation 6.20; however, its mean and variance can still be extracted from the information on the means and variances of the X_i's, giving only limited information on its randomness. If the functional relationship is linear, then the mean and variance of the response variable can be estimated without any approximation. For nonlinear functional relationships, the mean and variance of the response variable can only be estimated approximately. These are discussed next.

6.4.1 Partial Uncertainty Analysis: Response as a Linear Function of Multiple Random Variables

Linear relationships are discussed first. Suppose that

$$Y = aX_1 + bX_2 \tag{6.31}$$

where the X_i's have a mean of μ_{X_i} and standard deviation of σ_{X_i}, and a and b are constants. Although the exact distribution of Y is unknown, in general its mean and variance can be shown to be

$$\mu_Y = a\mu_{X_1} + b\mu_{X_2} \tag{6.32}$$

and

$$\mathrm{Var}(Y) = a^2\sigma_{X_1}^2 + b^2\sigma_{X_2}^2 + 2ab\mathrm{Cov}(X_1, X_2). \tag{6.33}$$

If X_1 and X_2 are statistically independent, then their covariance will be zero and Equation 6.33 becomes

$$\mathrm{Var}(Y) = a^2\sigma_{X_1}^2 + b^2\sigma_{X_2}^2. \tag{6.34}$$

In Equation 6.31, if

$$Y = aX_1 - bX_2 \tag{6.35}$$

then the mean and variance of Y can be calculated as

$$\mu_Y = a\mu_{X_1} - b\mu_{X_2} \tag{6.36}$$

and

$$\mathrm{Var}(Y) = a^2\sigma_{X_1}^2 + b^2\sigma_{X_2}^2 - 2ab\mathrm{Cov}(X_1, X_2). \tag{6.37}$$

In general, if

$$Y = \sum_{i=1}^{n} a_i X_i \tag{6.38}$$

where the X_i's are random variable with mean μ_{X_i} and standard deviation σ_{X_i} and the a_i's are constants, then the mean and variance of Y can be calculated as

$$\mu_Y = \sum_{i=1}^{n} a_i\mu_{X_i} \tag{6.39}$$

and

$$\begin{aligned}\mathrm{Var}(Y) &= \sum_{i=1}^{n} a_i^2\sigma_{X_i}^2 + \sum_{i}^{n}\sum_{j}^{n} a_i a_j \mathrm{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathrm{Cov}(X_i, X_j)\end{aligned} \tag{6.40}$$

since

$$\mathrm{Cov}(X_i, X_i) = \sigma_{X_i}^2 = \mathrm{Var}(X_i).$$

6.4.2 Approximate Solution: Response as a General Function of Multiple Random Variables

The concept developed in Section 6.4.1 for a linear relationship between the X_i's can be extended to any nonlinear relationship. Equation 6.1 will represent this case if in addition to the wind velocity, some or all of the variables in it are considered to be random. In general, the response variable Y can be represented by a relation g of a set of random variables as

$$Y = g(X_1, X_2, \ldots, X_n). \tag{6.41}$$

If the mean and variance of each X_i are known but the distribution is unknown, the approximate mean and variance of Y can be estimated, as discussed next.

Expanding the function $g(X_1, X_2, \ldots, X_n)$ in a Taylor series about the mean values $\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}$, one obtains

$$\begin{aligned}Y = g(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}) &+ \sum_{i=1}^{n}(x_i - \mu_{X_i})\frac{\partial g}{\partial X_i} \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - \mu_{X_i})(x_j - \mu_{X_j})\frac{\partial^2 g}{\partial X_i \partial X_j} + \ldots\end{aligned} \tag{6.42}$$

where the derivatives are evaluated at the mean values of the X_i's.

Truncating the series at the linear terms, the *first-order approximate mean* of Y, denoted as $E(Y')$, can be obtained as

$$E(Y') \approx g(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}) \tag{6.43}$$

which indicates that the first-order mean of Y is approximated by the value of the function evaluated at the mean values of the X_i's. The *first-order variance* of Y, denoted as $\mathrm{Var}(Y')$, can be shown to be

$$\mathrm{Var}(Y') \approx \sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i} \right) \mathrm{Var}(X_i) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{\partial g}{\partial X_i} \frac{\partial g}{\partial X_j} \mathrm{Cov}(X_i, X_j)$$

or

$$\mathrm{Var}(Y') \approx \sum_{i=1}^{n} E_i^2 \mathrm{Var}(X_i) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} E_i E_j \mathrm{Cov}(X_i, X_j)$$

or

$$\mathrm{Var}(Y') \approx \sum_{i=1}^{n} \sum_{j=1}^{n} E_i E_j \mathrm{Cov}(X_i, X_j), \tag{6.44}$$

where E_i and E_j are constants and are the values of the partial derivatives $\partial g / \partial X_i$ and $\partial g / \partial X_j$, respectively, evaluated at the mean values of the X_i's.

If the X_i's are uncorrelated, then Equation 6.44 reduces to

$$\mathrm{Var}(Y') \approx \sum_{i=1}^{n} E_i^2 \mathrm{Var}(X_i). \tag{6.45}$$

The coefficients E_i can be interpreted as amplification factors for the uncertainties in each of the corresponding random variables X_i. In general, these amplification factors will show the importance of the variables involved in the formulation. This type of probabilistic approach will also help to identify primary and secondary variables in problems where a large number of variables are involved.

This approximation of the mean and variance of Y can be improved by including the higher-order terms in the Taylor series expansion of $g(X_1, X_2, \ldots, X_n)$. If X_i and X_j are uncorrelated, the *second-order mean* of Y, denoted as $E(Y'')$, can be shown to be

$$E(Y'') \approx g\left(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}\right) + \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial^2 g}{\partial X_i^2} \right) \mathrm{Var}(X_i). \tag{6.46}$$

Again, the partial derivatives are evaluated at the mean values of all X_i's. To estimate the *second-order variance*, the information on the third and fourth moments of the X_i's must be available. However, in most cases this information will not be available. The use of the second-order mean and the first-order variance is considered adequate for most practical engineering applications.

EXAMPLE 6.9

Assume that the random variable Y can be represented by the following relationship:

$$Y = X_1 X_2^2 X_3^{1/3}$$

where X_1, X_2, and X_3 are statistically independent random variables with means of 1.0, 1.5, and 0.8, respectively, and corresponding standard deviations of 0.10, 0.20, and 0.15, respectively. Using Equations 6.43 and 6.45, we find the first-order mean and variance, respectively, to be

$$E(Y') \approx 1.0 \times 1.5^2 \times (0.8)^{1/3} = 2.0887$$

and

$$\begin{aligned}
\mathrm{Var}(Y') &\approx \mathrm{Var}(X_1)\left(\mu_{X_2}^2 \times \mu_{X_3}^{1/3}\right)^2 + \mathrm{Var}(X_2)\left[\mu_{X_1} \times (2\mu_{X_2}) \times \mu_{X_3}^{1/3}\right]^2 \\
&\quad + \mathrm{Var}(X_3)\left[\mu_{X_1} \times \mu_{X_2}^2 \times \left(\frac{1}{3}\mu_{X_3}^{-2/3}\right)\right]^2 \\
&= (0.10)^2(1.5^2 \times 0.8^{1/3})^2 + (0.20)^2(2 \times 1.0 \times 1.5 \times 0.8^{1/3})^2 \\
&\quad + (0.15)^2\left[1.0 \times 1.5^2 \times (1/3) \times 0.8^{-2/3}\right]^2 \\
&= (0.10)^2(2.09)^2 + (0.20)^2(2.78)^2 + (0.15)^2(0.87)^2 \\
&= 0.04363 + 0.31024 + 0.01704 = 0.37091
\end{aligned}$$

and $\sigma_Y = 0.609$.

In this example, the amplification factors E_1, E_2, and E_3 are found to be 2.09, 2.78, and 0.87, respectively, indicating the relative importance of each of the random variables.

Equation 6.46 can be used to calculate the second-order mean of Y as

$$\begin{aligned}
E(Y'') &\approx 2.0887 + \frac{1}{2}\mathrm{Var}(X_1) \times 0 + \frac{1}{2}\mathrm{Var}(X_2) \times \left(\mu_{X_1} \times 2.0 \times \mu_{X_3}^{1/3}\right) \\
&\quad + \frac{1}{2}\mathrm{Var}(X_3) \times \left[\mu_{X_1} \times \mu_{X_2}^2 \times \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right) \times \mu_{X_3}^{-5/3}\right] \\
&= 2.0887 + 0 + \frac{1}{2} \times (0.20)^2 \times \left(1.0 \times 2.0 \times 0.8^{1/3}\right) \\
&\quad + \frac{1}{2} \times (0.15)^2 \times \left[1.0 \times 1.5^2 \times \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right) \times 0.8^{-5/3}\right] \\
&= 2.0887 + 0.0371 - 0.0082 = 2.1176.
\end{aligned}$$

In this case, the second-order mean is slightly greater than the first-order mean.

EXAMPLE 6.10

The rate of steady water flow per time unit at a constant depth in a prismatic open channel can be calculated using the Manning formula as

$$Q = \frac{C_m}{n} A R^{2/3} S^{1/2}$$

where C_m is a constant of value 1 in SI units; n is the Manning roughness factor, which depends on the boundary materials being used; R is the hydraulic radius; and S is the slope of the bottom of the channel. Consider a trapezoidal cross section made with gravel. Assume C_m is a constant. However, n, A, R, and S are statistically independent random variables with means of 0.029, 8.0 m^2, 1.1 m, and 0.003, respectively, and the corresponding COVs are 0.30, 0.1, 0.1, and 0.1, respectively. No distributional information is available.

(a) Determine the first-order mean and variance of Q.

(b) Determine the second-order mean of Q.

SOLUTION

(a) Equation 6.43 is used to find the first-order mean of Q:

$$E(Q') \approx \frac{1}{0.029} \times 8.0 \times 1.1^{2/3} \times (0.003)^{1/2} = 16.10 \text{m}^3/\text{s}.$$

Equation 6.45 is used to find the first-order variance of Q:

$$\begin{aligned}
\text{Var}(Q') \approx{}& \text{Var}(n)\left(-\frac{1.0}{\mu_n^2} \times \mu_A \times \mu_R^{2/3} \times \mu_S^{1/2}\right)^2 \\
&+\text{Var}(A)\left(\frac{1.0}{\mu_n} \times \mu_R^{2/3} \times \mu_S^{1/2}\right)^2 + \text{Var}(R)\left(\frac{1.0}{\mu_n} \times \mu_A \times \frac{2}{3} \times \mu_R^{-1/3} \times \mu_S^{1/2}\right)^2 \\
&+\text{Var}(S)\left(\frac{1.0}{\mu_n} \times \mu_A \times \mu_R^{2/3} \times \frac{1}{2} \times \mu_S^{-1/2}\right)^2 \\
={}& (0.029 \times 0.3)^2\left(\frac{-1.0}{0.029^2} \times 8 \times 1.1^{2/3} \times 0.003^{1/2}\right)^2 \\
&+(8 \times 0.1)^2\left(\frac{1.0}{0.029} \times 1.1^{2/3} \times 0.003^{1/2}\right)^2 \\
&+(1.1 \times 0.1)^2\left(\frac{1.0}{0.029} \times 8 \times \frac{2}{3} \times 1.1^{-1/3} \times 0.003^{1/2}\right)^2 \\
&+(0.003 \times 0.1)^2\left(\frac{1.0}{0.029} \times 8 \times 1.1^{2/3} \times \frac{1}{2} \times 0.003^{-1/2}\right)^2 \\
={}& 23.33 + 2.59 + 1.15 + 0.65 = 27.72
\end{aligned}$$

or $\sigma'_Q \approx 5.26$ m^3/s.

(b) Equation 6.46 is used to find the second-order mean of Q:

$$E(Q'') \approx 16.10 + \frac{1}{2}\left[(0.029 \times 0.3)^2\left(\frac{2 \times 1}{0.029^3} \times 8 \times 1.1^{2/3} \times 0.003^{1/2}\right)\right.$$

$$+ (8\times 0.1)^2 \times 0$$
$$+ (1.1\times 0.1)^2\left(\frac{1}{0.029}\times 8\times\frac{2}{3}\times -\frac{1}{3}\times 1.1^{-4/3}\times 0.003^{1/2}\right)$$
$$+ (0.003\times 0.1)^2\left(\frac{1}{0.029}\times 8\times 1.1^{2/3}\times\frac{1}{2}\times\frac{-1}{2}\times 0.003^{-3/2}\right.$$
$$= 16.10 + \frac{1}{2}(2.898 + 0 - 0.036 - 0.040) = 17.51 \text{ m}^3/\text{s}.$$

6.5 MULTIPLE RANDOM VARIABLES WITH UNKNOWN RELATIONSHIP

In many cases, the exact form of g in Equation 6.41 may not be known. Consider a structure consisting of n members. The lateral deflection of the structure at the top is a function of each element's length, area, moment of inertia, Young's modulus, applied loads, and so forth; however, the exact functional relationship between the lateral deflection and all the other parameters is not known in most cases. A matrix analysis procedure can be used to estimate the lateral deflection for any specific values of these parameters. The exact functional relationship is known in algorithmic form but not in any exact functional form. The implication is that the partial derivatives of the function with respect to the random variables cannot be calculated to approximate the first- or second-order mean and the first-order variance of the response variable, as discussed in Section 6.4.2.

In this case, the approximate (first-order) mean value of the lateral deflection, represented by Y in Equation 6.41, can be obtained by analyzing the structure using the mean values of all the parameters in the problem, the same as in Equation 6.43. Evaluating the variance of Y will be more involved since the functional form of g is unknown, and its partial derivatives with respect to the ith random variable in Equation 6.41 cannot be evaluated. The task is to calculate the variance of Y without information on the analytical partial derivatives. The *Taylor series finite difference* (TSFD) estimation procedure can be used to numerically evaluate the variance of Y, as discussed below.

To evaluate the variance, the structure needs to be analyzed two more times, corresponding to each random variable:

$$Y_i^+ = g[\mu_{X_1}, \mu_{X_2}, \ldots, (\mu_{X_i} + \sigma_{X_i}), \ldots] \tag{6.47}$$

and

$$Y_i^- = g[\mu_{X_1}, \mu_{X_2}, \ldots, (\mu_{X_i} - \sigma_{X_i}), \ldots]. \tag{6.48}$$

In simple terms, Equation 6.47 states that the lateral displacement of the structure is calculated considering the mean values of all the random variables except the ith one, which is considered to be the mean plus one standard deviation value. Equation 6.48 indicates the same thing, except that for the ith random variable, the mean minus one standard deviation value needs to be considered. Then, using the central difference approximation, we can show that

$$E_i = \frac{\partial g}{\partial X_i} \approx \frac{Y_i^+ - Y_i^-}{2\sigma_{X_i}}. \tag{6.49}$$

Considering all the random variables, the first-order variance of Y is computed as

$$\mathrm{Var}(Y') \approx \sum_{i=1}^{n}\left(\frac{Y_i^+ - Y_i^-}{2\sigma_{X_i}}\right)^2 \quad \mathrm{Var}(X_i) \approx \sum_{i=1}^{n}\left(\frac{Y_i^+ - Y_i^-}{2}\right)^2. \tag{6.50}$$

Thus, when the functional relationship among the random variables is not known explicitly, the mean and variance of the response variable can be approximated by analyzing the problem two additional times for each of the random variables to be considered in the problem. If there are n random variables present in a problem, the total number of analyses required is $(1 + 2n)$.

EXAMPLE 6.11

To calculate the uncertainty in the wind pressure, consider Equation 6.1 again. For the sake of simplicity, K_{zt} and I are assumed to be constants with magnitudes of 1.0. V and K_z are assumed to be random variables with means of 63 mph and 0.68, respectively, and the corresponding coefficients of variation are 0.16 and 0.10. The standard deviations of V and K_z are 10.08 and 0.068, respectively.

Assuming that the functional relationship between the response variable q_z and the random variables V and K_z is known and is given by Equation 6.1, we can calculate the first-order mean of q_z as

$$E(q_z') \approx 0.00256 \times 0.68 \times 63^2 = 6.909 \text{ psf.}$$

The first-order variance of q_z can be shown to be

$$\mathrm{Var}(q_z') \approx \mathrm{Var}(K_z)\left(0.00256 \times \mu_V^2\right)^2 + \mathrm{Var}(V)\left(0.00256 \times \mu_{K_z} \times 2 \times \mu_V\right)^2$$

$$= (0.068)^2(0.00256 \times 63^2)^2 + (10.08)^2(0.00256 \times 0.68 \times 2 \times 63)^2 = 5.366.$$

To demonstrate the application of the TSFD, it is now assumed that the exact form of Equation 6.1 is not known. Even in this case, the mean value of q_z can be approximated by evaluating the equation at the mean values of all the random variables. For the problem under consideration, the mean of q_z will be 6.909 psf, as it was when the functional relationship was known. The partial of q_z with respect to the random variables cannot be calculated. However, using Equation 6.50, we can calculate the variance of q_z as described next.

For the random variable K_z, the mean plus one standard deviation and the mean minus one standard deviation values are (0.68 + 0.068 = 0.748) and (0.68 – 0.068 = 0.612), respectively. The corresponding values for V are 73.08 and 52.92 mph. Using Equations 6.47 and 6.48, we can obtain the following:

$$Y_{K_z}^+ = 0.00256 \times 0.748 \times 63^2 = 7.600$$

$$Y_{K_z}^- = 0.00256 \times 0.612 \times 63^2 = 6.218.$$

Similarly,

$$Y_V^+ = 0.00256 \times 0.68 \times 73.08^2 = 9.297$$

$$Y_V^- = 0.00256 \times 0.68 \times 52.92^2 = 4.875.$$

Thus, using Equation 6.50,

$$Var(q_z') \approx \left(\frac{7.600 - 6.218}{2}\right)^2 + \left(\frac{9.297 - 4.875}{2}\right)^2 = 5.366.$$

The same value for the variance of q_z was estimated when the functional relationship was known. However, in this case, partial derivatives were numerically calculated using Equation 6.49. In reality, for complicated problems, the numerical values of the response variable may need to be calculated with a computer program or other numerical techniques for different combinations of values for all the random variables. But this simple example clearly demonstrates that the variance of the response variable can be estimated even when the functional relationship among the random variables is not known explicitly.

6.6 REGRESSION ANALYSIS

In the previous sections we discussed analytical procedures for calculating uncertainties in the response variable when it is a function of other basic random variables, either explicit (known) or implicit (analytically unknown but known in an algorithmic sense, e.g., in the form of a finite element representation). When the functional relationship between the response and the basic random variables is implicit or unknown, it can be developed by considering the available information on them. Since some or all of the variables are random, the relationship will be probabilistic in nature. *Regression analysis*, a statistical technique that can be used to investigate and model this probabilistic relationship, is discussed in this section.

For example, in geotechnical engineering, the looseness or denseness of a cohesionless soil deposit is usually characterized by its relative density. The relative density, D_r, is defined as

$$D_r = \frac{\gamma_{max}}{\gamma} \frac{\gamma - \gamma_{min}}{\gamma_{max} - \gamma_{min}} \tag{6.51}$$

where γ_{max}, γ_{min}, and γ are the maximum, minimum, and in-place dry density, respectively, of a cohesionless deposit. The basic drawback of this definition is that it is computed from the ratio of small differences between large numbers. This implies that small variations in the density could lead to large errors in the computed relative density. The uncertainties in D_r in terms of its coefficient of variation could be on the order of 0.36, which is not negligible.

The standard penetration test can be used to measure the degree of compactness of in situ soil. The standard penetration resistance, known as the SPT-value or N value, has been used since 1948 to estimate the in situ relative density of a cohesionless soil deposit. N is defined as the number of blows required to drive a split-barrel sampler the last 12 inches of an 18-inch drive using a 140-pound hammer falling 30 inches. Due to the nature of the testing procedure and the field environment, the N values obtained from the field will contain a considerable amount of uncertainty. For a homogeneous soil deposit, a coefficient of variation of 0.25 for N is reported in the literature. Since the N values and D_r measure the same soil property, there must be some relationship between them. Since both are random variables, regression analysis can be used to determine the relationship.

Determining the linear relationship between two variables is called *simple linear regression* analysis. Determining the linear relationship between more than two vari-

ables is called *multiple linear regression* analysis. Determining a nonlinear relationship between random variables is called *nonlinear regression* analysis. The basic concept behind regression analysis is discussed briefly in the following sections.

6.6.1 Simple Linear Regression Analysis

Suppose n pairs of data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are available for two variables X and Y. To obtain the relationship between them, the data can be plotted as shown in Figure 6.1. This graph is known as a *scatter diagram* and provides a variety of information, including whether the relationship is linear or nonlinear and whether the spread in the data is uniform or constant about the average tendency. For a given value of x, y will have a range of values. This can be expressed mathematically as

$$Y = b_0 + b_1 X + \varepsilon. \tag{6.52}$$

In general, X is called an independent or *predictor* or *regressor* variable, and Y is called the dependent or *response* variable. b_0 and b_1 represent the intercept and slope of the line and are known as *regression coefficients*. ε, representing the error in the estimation of Y, is a random variable representing the differences between the observed and predicted values using the regression equation of the dependent variable. The form of Equation 6.52 suggests that it is the *linear regression* of Y on X. Obviously, the linear regression of X on Y is also possible.

In the context of linear regression analysis, Equation 6.52 is generally expressed as

$$E(Y \mid X = x) = E(Y \mid x) = b_0 + b_1 x. \tag{6.53}$$

Equation 6.53 gives the expected value of Y for a given value of $X = x$. The task is to calculate the regression coefficients using the data on X and Y. Some of the underlying assumptions in estimating the regression coefficients are (a) the scatter diagram indicates that the relationship between X and Y can be approximated by a straight line; (b) the error term ε has a zero mean and an unknown but constant variance of σ^2, implying that the spread in the data about the regression line is constant; and (c) errors are uncorrelated and normally distributed. A regression model will be unacceptable if these assumptions are not satisfied.

The most appropriate regression line can be estimated by minimizing the sum of squares of the differences (errors) between the observed and predicted values of the

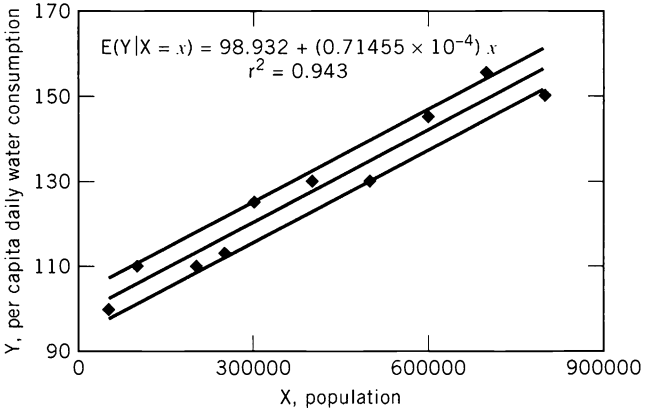


Figure 6.1 Scatter Diagram

dependent variable. This is usually done using the *method of least squares*. If the predicted value of the dependent variable for the ith observation is $\hat{y}_i$, the *error sum of squares* (SS_E) for n observations is

$$SS_E = \varepsilon^2 = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}(y_i - b_0 - b_1 x_i)^2. \tag{6.54}$$

With the method of least squares, the two equations for the two unknown regression coefficients are

$$\frac{\partial \varepsilon^2}{\partial b_0} = \sum_{i=1}^{n} 2(y_i - b_0 - b_1 x_i)(-1) = 0$$

and

$$\frac{\partial \varepsilon^2}{\partial b_1} = \sum_{i=1}^{n} 2(y_i - b_0 - b_1 x_i)(-x_i) = 0.$$

If these two equations are solved simultaneously, the estimates of the two regression coefficients, $\hat{b}_0$, and $\hat{b}_1$, can be shown to be

$$\hat{b}_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{\hat{b}_1}{n}\sum_{i=1}^{n} x_i = \bar{y} - \hat{b}_1 \bar{x} \tag{6.55}$$

$$\hat{b}_1 = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2} = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} \tag{6.56a}$$

where $\bar{x}$ and $\bar{y}$ are the estimated mean values of X and Y. Equation 6.56a can also be conveniently expressed as

$$\hat{b}_1 = \frac{\sum_{i=1}^{n} y_i(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{S_{xy}}{S_{xx}}. \tag{6.56b}$$

Thus, the mean least-squares regression is

$$E(Y \mid X = x) = \hat{b}_0 + \hat{b}_1 x. \tag{6.57}$$

The regression equation is applicable only over the range of observed data of the independent variable X. A regression equation gives a measure of the mean relationship between variables. Obviously, the dispersion about the mean equation, known as the *conditional variance* Var $(Y \mid X = x)$ or Var $(Y \mid x)$ or the *error mean square* or *residual mean square* (MS_E), is also of interest for checking the adequacy of the regression equation. The unbiased conditional variance can be estimated as

$$\begin{aligned} MS_E = \text{Var}(Y \mid X = x) = S^2_{Y|x} &= \frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 \\ &= \frac{\varepsilon^2}{n-2} = \frac{SS_E}{n-2}. \end{aligned} \tag{6.58}$$

The corresponding *conditional standard deviation*, $S_{Y \mid x}$, is known as the *standard error of regression*.

6.6.2 Coefficient of Determination

The adequacy of a proposed regression model in representing the relationship between the regressor and the response variables is a major concern. One of many parameters that can be used for such determination is the *coefficient of determination* R^2.

The total variability in the response variable Y, denoted as S_{yy}, has two components: the amount of variability in the observations of the y_i's accounted for by the regression model (SS_R), and the residual variation left unexplained by the regression model (SS_E). Mathematically, the concept can be represented as

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 \tag{6.59a}$$

or

$$S_{yy} = SS_R + SS_E. \tag{6.59b}$$

A nondimensional quantity R^2, called the coefficient of determination, can be defined as

$$R^2 = \frac{SS_R}{S_{yy}} = 1 - \frac{SS_E}{S_{yy}} = \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}. \tag{6.60}$$

R^2 will have a value between 0 and 1. When it is close to 1, it implies that most of the variability in the dependent variable Y is explained by the regression model.

The information on R^2 should be used with caution, since it is always possible to increase its value. By adding more regressor variables, the R^2 value can be increased. However, the new model is not necessarily better than the old model. The R^2 value also depends on the range of the regressor variable; it will increase as the spread of the regressor variable increases and decrease as the spread in the regressor variable decreases. Also, a large R^2 value does not imply that the regression equation is an accurate predictor. Montgomery and Peck (1982) presented cases where the relationships between the regressor and dependent variables were nonlinear but a linear model gave a large R^2 value. Some of the other important issues on regression analysis are briefly discussed in Section 6.6.3.

EXAMPLE 6.12

Water consumption, expressed in gallons per capita per day, is a major concern to city administrators in a growing desert city. To address the problem comprehensively, 10 cities of different population sizes in the Southwest were studied. The results are summarized in Table 6.1; X denotes the population size and Y denotes water consumption in gallons per capita per day. The scatter diagram in Figure 6.1 indicates that a linear relationship exists between X and Y.

(a) Determine the regression equation of Y on X.

(b) Determine the conditional variance and conditional standard deviation.

(c) Determine the coefficient of determination.

(d) Suppose the population of the city will be 750,000 in 2010. If the water consumption corresponding to a given population is a normal random variable, what is the probability that the per capita water consumption will exceed 150 gallons per day?

Table 6.1 Example on Linear Regression

City	x_i ($\times 10^5$)	y_i	$x_i y_i$ ($\times 10^6$)	x_i^2 ($\times 10^{10}$)	y_i^2 ($\times 10^4$)	$\hat{y}_i$	$y_i - \hat{y}_i$	$SS_E = (y_i - \hat{y}_i)^2$	$S_{yy} = (y_i - \bar{y})^2$
1	0.5	100	5.00	0.25	1.0000	102.505	−2.505	6.2739	718.24
2	1.0	110	11.00	1.00	1.2100	106.078	3.922	15.3830	282.24
3	2.0	110	22.00	4.00	1.2100	113.223	−3.223	10.3877	282.24
4	2.5	113	28.25	6.25	1.2769	116.796	−3.796	14.4077	190.44
5	3.0	125	37.50	9.00	1.5625	120.369	4.632	21.4508	3.24
6	4.0	130	52.00	16.00	1.6900	127.514	2.486	6.1802	10.24
7	5.0	130	65.00	25.00	1.6900	134.660	−4.660	21.7109	10.24
8	6.0	145	87.00	36.00	2.1025	141.805	3.195	10.2080	331.24
9	7.0	155	108.50	49.00	2.4025	148.951	6.050	36.5965	795.24
10	8.0	150	120.00	64.00	2.2500	156.096	−6.096	37.1612	538.24
Σ	39.0	1,268	536.25	210.50	16.3944			179.7630	3,161.60

SOLUTION

(a) Observing the trend in the scatter diagram in Figure 6.1, we consider for the problem a linear regression equation of the form given by Equation 6.53. Using Equations 6.55 and 6.56a, and the numbers summarized in Table 6.1, we can calculate the two regression coefficients $\hat{b}_0$ and $\hat{b}_1$ as

$$\hat{b}_1 = \frac{536.25 \times 10^6 - \frac{1}{10} \times 39 \times 10^5 \times 1,268}{210.5 \times 10^{10} - \frac{1}{10} \times (39 \times 10^5)^2} = 0.71455 \times 10^{-4}$$

and

$$\hat{b}_0 = \frac{1}{10} \times 1,268 - \frac{0.71455 \times 10^{-4}}{10} \times 39 \times 10^5 = 98.932.$$

Thus, the regression equation becomes

$$E(Y \mid X = x) = 98.932 + 0.71455 \times 10^{-4} x.$$

(b) Using the information from Table 6.1, we find SS_E to be 179.763. Using Equation 6.58, we can calculate the conditional variance as

$$S^2_{Y|x} = \frac{179.7630}{10 - 2} = 22.470.$$

The corresponding conditional standard deviation is

$$S_{Y|x} = \sqrt{22.470} = 4.740.$$

(c) Using the information from Table 6.1, we can observe that $S_{yy} = 3,161.60$ (note the mean value of y is 126.8), and $SS_E = 179.763$. Thus, using Equation 6.60, we find the coefficient of determination to be

$$R^2 = 1 - \frac{179.763}{3161.60} = 0.943.$$

The scatter diagram shown in Figure 6.1 and the estimated R^2 value indicate that the predictability of the regression equation is quite high.

(d) Using the regression equation, the mean per capita water consumption in 2010 will be

$$E(Y \mid X = 750,000) = 98.932 + 0.71455 \times 10^{-4} \times 750,000 = 152.523.$$

Thus

$$P(Y > 150 \mid X = 750,000) = 1 - \Phi\left(\frac{150 - 152.523}{4.740}\right)$$
$$= 1 - \Phi(-0.53) = \Phi(0.53) = 0.70194.$$

6.6.3 Residual Analysis

The underlying assumptions for the preceding regression analysis were identified in Section 6.6.2. Before a regression model is accepted, it is important to evaluate the extent to which these assumptions are satisfied.

This is done using *residual analysis*. The residual, ε_i, in a regression analysis is usually defined as the observed value of the dependent variable minus the predicted value and can be expressed as

$$\varepsilon_i = y_i - \hat{y}_i. \tag{6.61}$$

Physically, the residual is a measure of the variability in the dependent variable Y not accounted for by the regression model. Thus, any local deviations from the assumptions made in the regression model will show up in the residuals.

Though several statistical techniques are available, residual plotting is probably the most informative for residual analysis. The residuals can be plotted on normal probability paper, as discussed in Section 5.2.1.1. If the plot appears to be linear, it will satisfy the normality assumption of the residuals. Montgomery and Peck (1982) noted that for sample size $n \leq 16$, the normal probability plots may deviate substantially from linearity. Also, the normal probability plots often exhibit no unusual behavior even if the errors ε_i are not normally distributed.

A considerable amount of information can be obtained by plotting the residuals versus the predicted values $\hat{y}_i$ of the dependent variable. If the residuals have a horizontal band on both sides of the zero axis, as shown in Figure 6.2, the mean value of the

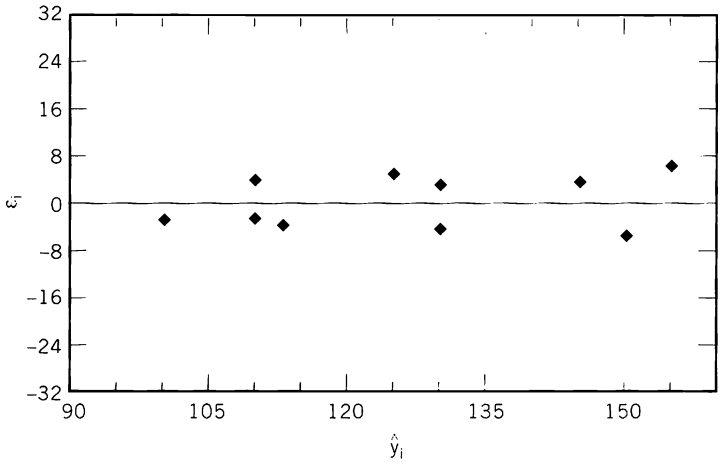


Figure 6.2 Residual Analysis of Data Shown in Figure 6.1

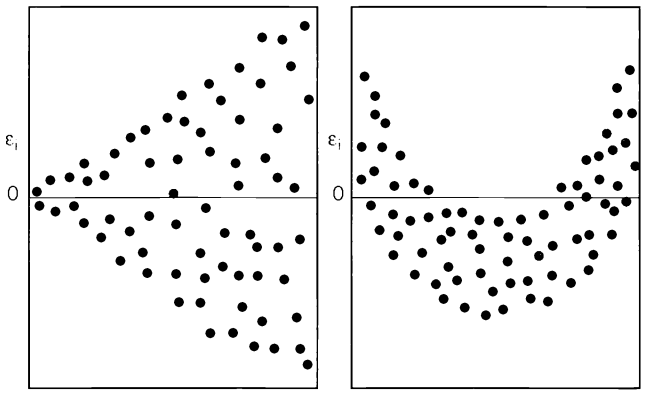


Figure 6.3 Residual Analysis, (a) Nonconstant Variance; (b) Nonlinear Relationship

residual is approximately zero, and the variance is constant about the regression line. If the residuals diverge as shown in Figure 6.3a, the variance is not constant, thus invalidating the regression analysis. Weights on independent variables can be used to address the nonconstant variance problem (Ang and Tang, 1975). If the residuals appear as shown in Figure 6.3b, the linear regression analysis is unsatisfactory; higher order terms may be necessary, for example, a square term or some other transformation of the independent variable. The residuals can also be plotted against the independent variable X, providing similar information.

The point is that linear regression analysis may appear to be simple, but in reality it is not. Several underlying assumptions need to be satisfied before a regression equation can be accepted.

EXAMPLE 6.13

Determine whether the assumptions of regression analysis are satisfied for the water consumption problem considered in Example 6.11.

SOLUTION

Since the normal probability plot is not expected to be informative for a relatively small sample size, in this case $n = 10$, it is not considered here. However, Figure 6.2 indicates that the variance is constant about the regression line. Also, the residuals are evenly distributed over both sides of the zero axis, indicating that its mean is expected to be 0. In fact, for this particular example, the calculated value of the mean of the residuals is 0. Thus, the example satisfies the assumptions of regression analysis.

EXAMPLE 6.14

Composite materials are now being used in structural design. It is known that the stress–strain properties depend on many factors including the types of fiber and matrix

used; the orientation, volume ratio, and layering of fibers; and manufacturing procedures. To determine the Young's modulus of a batch of composite material, a cross section with an area of 0.5 in.^{2} and length of 8 inches is subjected to a monotonically increasing tensile load in the laboratory, and the corresponding elongation of the rod is noted. The results are shown in Table 6.2. It is necessary to estimate the Young's modulus of the material using these observations.

SOLUTION

For the observed data, the stress (Y) can be calculated as load/area, and the strain (X) can be calculated as elongation/8. Assuming the stress–strain diagram to be linear in the observed ranges of data, and noting that when the stress is zero, the strain will also be zero, we can consider the following linear regression model (without intercept):

$$E(Y \mid X = x) = bx. \tag{6.62}$$

In this case, the sum of the squares of the differences for n observations becomes

$$SS_E = \varepsilon^2 = \sum_{i=1}^{n} (y_i - bx_i)^2. \tag{6.63}$$

The least-squares estimate of the slope can be calculated using

$$\frac{\partial \varepsilon^2}{\partial b} = \sum_{i=1}^{n} 2(y_i - bx_i)(-x_i) = 0$$

or the estimated slope ($\hat{b}$) of the regression line can be shown to be

$$\hat{b} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}. \tag{6.64}$$

The estimated slope of the regression line is the Young's modulus of the composite material.

Table 6.2 Young's Modulus Calculation

No.	Load (kip)	Elongation ($\times 10^{-3}$in.)	Strain = elongation/8 x_i ($\times 10^{-4}$)	Stress = load/area y_i (ksi)	x_i^2 ($\times 10^{-8}$)	$x_i y_i$ ($\times 10^{-4}$)	$\hat{y}_i$ (ksi)	$SS_E = (y_i - \hat{y}_i))^2$ (10^{-3})	$S_{yy} = (y_i - \bar{y})^2$
1	0.5	1.9	2.375	1.0	5.641	2.375	0.936	4.096	3.331
2	0.7	2.9	3.625	1.4	13.141	5.075	1.429	0.841	2.031
3	1.0	4.2	5.250	2.0	27.563	10.500	2.070	4.900	0.681
4	1.3	5.0	6.250	2.6	39.063	16.250	2.464	18.496	0.051
5	1.5	6.0	7.500	3.0	56.250	22.500	2.957	1.849	0.031
6	1.8	7.5	9.375	3.6	87.891	33.750	3.697	9.409	0.601
7	2.0	7.9	9.875	4.0	97.516	39.500	3.894	11.236	1.381
8	2.5	10.3	12.875	5.0	165.766	64.375	5.077	5.929	4.731
Σ					492.831	194.325		56.756	12.838

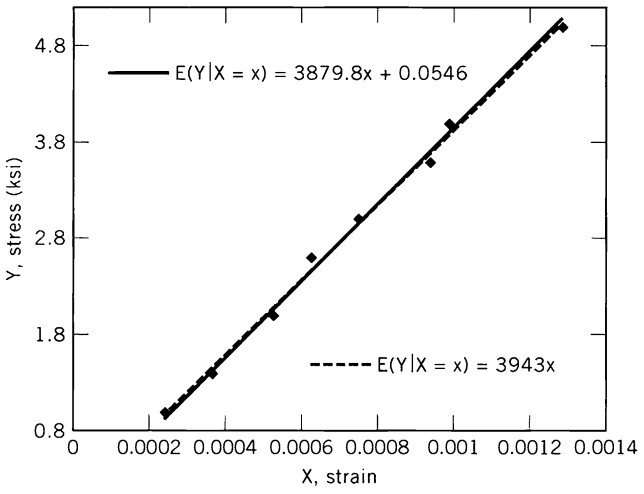


Figure 6.4 Data on Stress and Strain With and Without Intercept

All the calculations necessary to estimate the Young's modulus using Equation 6.64 are summarized in Table 6.2. The Young's modulus in this case is

$$\hat{b} = \frac{194.325 \times 10^{-4}}{492.831 \times 10^{-8}} = 3,943 \text{ ksi.}$$

The corresponding regression equation is

$$E(Y \mid X = x) = 3,943x.$$

Using the information from Table 6.2, we can observe that $S_{yy} = 12.838$ (note the mean value of y is 2.825 ksi), and $SS_E = 56.756 \times 10^{-3}$. Thus, using Equation 6.60, we find the coefficient of determination to be

$$R^2 = 1 - \frac{SS_E}{S_{yy}} = 1 - \frac{56.756 \times 10^{-3}}{12.838} = 0.996.$$

If the intercept b_0 is not assumed to be zero, then following the same procedures discussed in the previous example, and using eight sets of observations on X and Y given in Table 6.2, we can obtain the following regression equation:

$$E(Y \mid X = x) = 0.0546 + 3879.8x.$$

The R^2 value is found to be 0.995. Thus, both regression equations, one without the intercept and the other with intercept, have high R^2 values, but the corresponding Young's modulus values are 3,943 and 3,879.8 ksi, respectively. Both regression equations are plotted in Figure 6.4, and they appear to be similar. However, considering the physical aspects of the problem, the equation without the intercept is more appropriate in this case.

6.6.4 Multiple Linear Regression

The discussion on linear regression analysis in the previous section can be generalized to consider multiple independent variables, $X_1, X_2, \ldots, X_m$. Similar to Equation 6.53, the mean value of the regression equation can be expressed as

$$E(Y \mid x_1, x_2, \ldots, x_m) = b_0 + b_1 x_1 + \ldots + b_m x_m \tag{6.65}$$

where b_0, b_1, b_2, ..., b_m are regression coefficients or *partial regression coefficients*; b_j represents the change in the mean value of the dependent variable Y for a unit change in x_j. The partial regression coefficients need to be estimated using the available data. As before, the basic assumption is that the conditional variance $\mathrm{Var}(Y \mid x_1, x_2, \ldots, x_m)$ is a constant. Similar to Equation 6.54, the total sum of squared errors in this case is

$$SS_E = \varepsilon^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i-1}^{n} \left(y_i - b_0 - \sum_{j=1}^{m} b_j x_{ij} \right)^2. \tag{6.66}$$

Using the least-squares principle, minimizing ε^2 with respect to the b_i's will result in $(m + 1)$ number of equations. Solving these equations simultaneously will give the estimate of all the partial regression coefficients.

Once the regression coefficients are evaluated, ε^2 can be estimated using Equation 6.66. An unbiased estimate of the conditional variance can be calculated as

$$S^2_{Y|x_1, x_2, \ldots, x_m} = \frac{\varepsilon^2}{n - m - 1} \tag{6.67}$$

where n is the number of sets of data, and m is the number of independent variables. Using Equation 6.60, we can also calculate the corresponding coefficient of determination, R^2.

The algebraic method of minimizing ε^2 and solving $(m + 1)$ simultaneous equations is quite cumbersome for a large number of variables (m) and a large number of data sets (n). A more compact method is to use matrix notation, as discussed next.

For each regressor variable X_i, n sets of data are collected, giving the total number of sample points $n \times m$: in other words, each data set consists of m observations representing m regressor variables. The general form of a regression model in the matrix notation can be expressed as

$$\mathbf{Y} = \mathbf{Xb} + \varepsilon \tag{6.68}$$

or

$$\begin{Bmatrix} Y_1 \\ Y_2 \\ \cdots \\ \cdots \\ \cdots \\ Y_n \end{Bmatrix} = \begin{Bmatrix} 1 & x_{11} & x_{21} & \cdots & \cdots & \cdots & x_{m1} \\ 1 & x_{12} & x_{22} & \cdots & \cdots & \cdots & x_{m2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{1n} & x_{2n} & \cdots & \cdots & \cdots & x_{mn} \end{Bmatrix} \begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdots \\ \cdots \\ \cdots \\ b_m \end{Bmatrix} + \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \cdots \\ \cdots \\ \varepsilon_n \end{Bmatrix}. \tag{6.69}$$

where $\mathbf{Y}$ is a vector of size $(n \times 1)$ containing n observations of the dependent variable, $\mathbf{X}$ is a matrix of size $n \times (m + 1)$, x_{ij} is the observation on the ith regressor variable in the jth data set, $\mathbf{b}$ is a vector of size $(m + 1) \times 1$ containing all the partial regression coefficients to be determined, and ε is a vector of size $(n \times 1)$ containing the errors or residuals. ε_i is similar to ε in Equation 6.52 for linear regression; the ε_i's are uncorrelated and identically distributed normal random variables with zero mean and a variance of σ^2.

The corresponding regression equation is

$$\mathbf{E}(\mathbf{Y} \mid \mathbf{X} = \mathbf{x}) = \mathbf{xb}. \tag{6.70}$$

Equations 6.68 and 6.69 are simply vector addition formulas, as shown in Figure 6.5. As mentioned earlier, the method of least-squares fit minimizes the sum of squares of

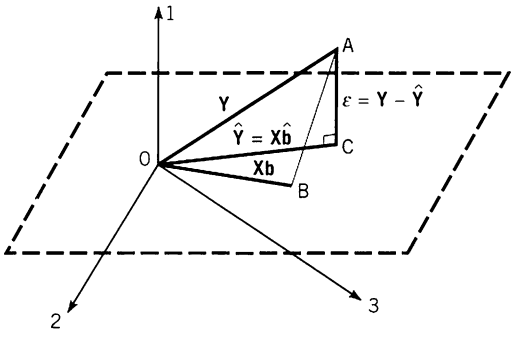


Figure 6.5 Vector Representation of the Principles of Least Squares

the errors. With reference to Figure 6.5, the length of the residual vector ε is defined as

$$|\varepsilon| = \sqrt{\sum_{i=1}^{n} \varepsilon_i^2}. \tag{6.71}$$

Therefore, minimizing the sum of squares of the errors (the right-hand side of Equation 6.71) would also minimize the length of the residual vector ε (the left-hand side of Equation 6.71). From Figure 6.5, it is clear that the minimum length of the vector ε is achieved when ε is perpendicular to the vector $\mathbf{Xb}$, that is, when the dot product of the two vectors is zero. This condition is written as

$$\mathbf{Xb} \cdot \varepsilon = 0 \tag{6.72}$$

or

$$[\mathbf{Xb}]' \varepsilon = 0$$

$$\mathbf{b}^t\mathbf{X}^t(\mathbf{Y} - \mathbf{Xb}) = 0$$

$$\mathbf{b}^t\mathbf{X}^t\mathbf{Y} = \mathbf{b}^t\mathbf{X}^t\mathbf{Xb}.$$

Here, the superscript t refers to the transpose of a matrix. Therefore,

$$\mathbf{b} = [\mathbf{X}^t\mathbf{X}]^{-1}[\mathbf{X}^t\mathbf{Y}]. \tag{6.73}$$

This is a well-known formula to obtain the coefficients of the multiple linear regression model. Note that the matrices $\mathbf{X}$ and $\mathbf{Y}$ are available from observed data, as defined in Equation 6.69. The matrix multiplication and inversion routines are readily available in many computer programs and spreadsheets.

Once the regression coefficients are available, the SS_E, $S^2_{Y|x_1,x_2,\ldots,x_m}$, and R^2 values can be calculated using Equations 6.66, 6.67, and 6.60, respectively.

EXAMPLE 6.15

Trichlorophenol (TCP), a nonbiodegradable organic compound, can be removed by chemical oxidation using an oxidant such as hydrogen peroxide, with ferrous sulfate as a catalyst. The ratio of the concentration of TCP after the oxidation to the concentration of TCP before the oxidation is denoted as Y in this example and is affected by four variables: temperature (X_1), acidity measured by pH (X_2), concentration of the oxidant (X_3), and concentration of the catalyst (X_4). The available data from 12 different experiments are shown in the following table:

X_1 (°C)	X_2	X_3 (mg/L)	X_4 (mg/L)	Y
20	3.5	1000	50	0.53
10	3.8	500	40	0.80
15	4.0	600	25	0.82
30	2.7	1200	100	0.26
25	4.2	1800	75	0.25
18	2.5	900	80	0.50
27	3.0	700	30	0.69
22	3.2	1500	150	0.19
32	3.7	450	130	0.37
16	2.4	1000	90	0.47
20	3.9	800	60	0.54
35	4.5	1800	100	0.15

(a) Fit a multiple linear regression model to these data.

(b) Determine the R^2 value for the regression model.

SOLUTION

(a) The linear regression model to represent the four regressor variables can be expressed as

$$E(Y \mid x_1, x_2, x_3, x_4) = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4$$

The **X** matrix in Equation 6.70 is

$$\mathbf{X} = \begin{bmatrix} 1 & 20 & 3.5 & 1000 & 50 \\ 1 & 10 & 3.8 & 500 & 40 \\ 1 & 15 & 4.0 & 600 & 25 \\ 1 & 30 & 2.7 & 1200 & 100 \\ 1 & 25 & 4.2 & 1800 & 75 \\ 1 & 18 & 2.5 & 900 & 80 \\ 1 & 27 & 3.0 & 700 & 30 \\ 1 & 22 & 3.2 & 1500 & 150 \\ 1 & 32 & 3.7 & 450 & 130 \\ 1 & 16 & 2.4 & 1000 & 90 \\ 1 & 20 & 3.9 & 800 & 60 \\ 1 & 35 & 4.5 & 1800 & 100 \end{bmatrix}$$

The **Y** vector is

$$\mathbf{Y}^t = \{0.53 \;\; 0.80 \; 0.82 \; 0.26 \; 0.25 \; 0.50 \; 0.69 \; 0.19 \; 0.37 \; 0.47 \; 0.54 \; 0.15\}$$

The quantities $\mathbf{X}^t\mathbf{X}$ and $\mathbf{X}^t\mathbf{Y}$ are calculated as:

$$\mathbf{X}^t\mathbf{X} = \begin{bmatrix} 12 & 270 & 41.4 & 12250 & 930 \\ 270 & 6692 & 942.7 & 292500 & 22500 \\ 41.4 & 942.7 & 148.02 & 43035 & 3163 \\ 12250 & 292500 & 43035 & 14922500 & 1034500 \\ 930 & 22500 & 3163 & 1034500 & 88750 \end{bmatrix}$$

and

$$\mathbf{X}'\mathbf{Y} = \begin{bmatrix} 5.570 \\ 112.170 \\ 19.133 \\ 4740.500 \\ 350.750 \end{bmatrix}$$

The regression coefficients are obtained as

$$\begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix} = [\mathbf{X}'\mathbf{X}]^{-1}[\mathbf{X}'\mathbf{Y}] = \begin{Bmatrix} 1.06619 \\ -0.00760 \\ 0.01080 \\ -0.00024 \\ -0.00288 \end{Bmatrix}$$

Thus, the regression equation is

$$E(Y \mid x_1, x_2, x_3, x_4) = 1.06619 - 0.00760x_1 + 0.01080x_2 - 0.00024x_3 - 0.00288x_4 .$$

(b) The coeffecient of determination R^2 is found using Equation 6.60. In this equation, $\bar{y}$, the average of the observed values of Y, is found to be 0.4642. The predicted values, $\hat{y}_i$, are computed from the regression equation for each of the 12 data sets, and the observed values, y_i, are readily available from the data. Thus R^2 is calculated as

$$R^2 = \frac{\sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n} (y_i - \bar{y})^2} = \frac{0.55918}{0.57209} = 0.977.$$

The coefficient of determination is high, indicating that the regression equation models the observed data quite well. However, in all cases, the analyst must ensure that the model is also physically meaningful.

6.6.5 Nonlinear Models

The procedure just discussed for multiple linear regression can also be used to fit polynomial models and other nonlinear models that are similar to linear models. Consider a polynomial model as

$$E(Y \mid X = x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m. \tag{6.74}$$

Each of the powers of X can simply be considered to be a different variable, for example, $X_1 = X$, $X_2 = X^2$, and so on, and the same multiple regression formula used in Equation 6.73 can be used to determine the coefficients of the polynomial model.

Other nonlinear models such as exponential and power models can also be transformed to linear form to facilitate the use of linear regression. Consider the exponential model as

$$E(Y \mid X = x) = b_0 \exp(b_1 x). \tag{6.75}$$

Taking the natural logarithm on both sides results in

$$E(\ln Y \mid X = x) = \ln b_0 + b_1 x. \tag{6.76}$$

Since ln b_0 can be considered to be another regression coefficient, Equation 6.76 represents a linear regression model as discussed in Section 6.6.1. Here, the independent variable is X and the dependent variable is ln Y.

EXAMPLE 6.16

The initial bacterial concentration of a pure culture suspended in water needs to be estimated. It is known from previous experience that the bacteria grow in an exponential form, given by Equation 6.75. Every 30 minutes, the number of cells in the culture is calculated, and the results are given as follows:

Time (hour)	Concentration ($\times 10^5$ cell/mL)
0.5	1.80
1.0	1.85
1.5	2.25
2.0	4.53
2.5	5.34
3.0	9.50
3.5	13.50
4.0	18.50

(a) Calculate the expected initial bacterial concentration of the culture.

(b) Calculate the coefficient of determination of the regression equation.

SOLUTION

The scatter diagram of number of bacteria N_B versus time T is plotted in Figure 6.6a. Similar to Equation 6.75, the following regression equation is selected:

$$E(N_B \mid T = t) = b_0 \exp(b_1 t).$$

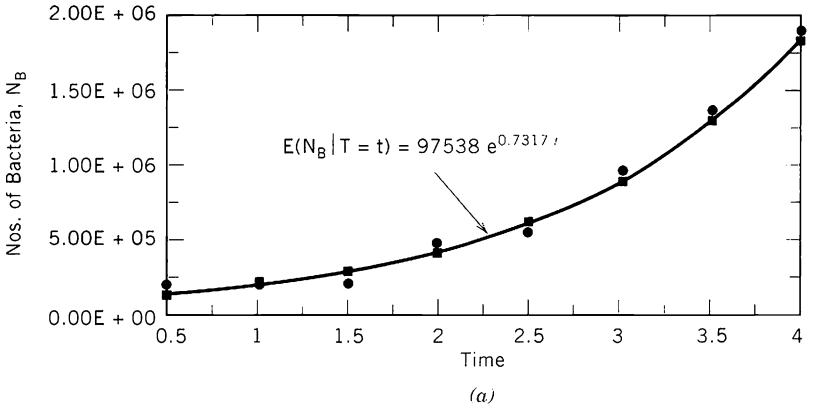


Figure 6.6a N_B Versus Time

Taking the natural logarithm of both sides results in

$$E(\ln N_B \mid T = t) = \ln b_0 + b_1 t = b_0' + b_1 t.$$

Thus, a linear regression analysis between $\ln N_B$ and T is carried out and the results are summarized in Table 6.3. Equations 6.55 and 6.56a can be used to estimate the two regression coefficients $\hat{b}_0'$ and $\hat{b}_1$:

$$\hat{b}_1 = \frac{244.102 - \frac{1}{8} \times 18 \times 105.075}{51.00 - \frac{1}{8} \times (18.0)^2} = 0.7317$$

and

$$\hat{b}_0' = \frac{1}{8} \times 105.075 - \frac{0.7317}{8} \times 18 = 11.488.$$

Thus, the regression equation becomes

$$E(\ln N_B \mid T = t) = 11.488 + 0.7317t.$$

This is shown in Figure 6.6b. The regression equation can also be expressed as

$$E(N_B \mid T = t) = e^{11.488 + 0.7317t} = 97{,}538e^{0.7317t}.$$

The initial concentration of bacteria at $t = 0$ is 97,538 cell/mL.

(b) Using the information given in Table 6.3, we can calculate the coefficient of determination of the regression equation to be

$$R^2 = 1 - \frac{0.1710}{5.7930} = 0.970.$$

Consider next the regression of a power model of the form

$$E(Z \mid X = x, Y = y) = ax^b y^c. \tag{6.77}$$

Again, by taking natural logarithms on both sides, Equation 6.77 becomes a linear model of the form

$$E(\ln Z \mid X = x, Y = y) = \ln a + b \ln x + c \ln y. \tag{6.78}$$

Equation 6.78 represents a multiple linear regression model as discussed in Section 6.6.4. In this case, the regressor variables are $\ln X$ and $\ln Y$, and the dependent variable is $\ln Z$.

Table 6.3 Regression Analysis on Number of Bacteria Versus Time

No.	t_i (hours)	n_{B_i} ($\times 10^5$)	$n'_{B_i} = \ln n_{B_i}$	$t_i n'_{B_i}$	t_i^2	$n'^2_{B_i}$	$\hat{n}'_{B_i}$	$(n'_{B_i} - \hat{n}'_{B_i})^2$	$(n'_{B_i} - \bar{n}'_B)^2$
1	0.5	1.80	12.1007	6.0504	0.25	146.4272	11.8538	0.0609	1.0685
2	1.0	1.85	12.1281	12.1281	1.00	147.0911	12.2197	0.0084	1.0126
3	1.5	2.25	12.3239	18.4858	2.25	151.8774	12.5856	0.0685	0.6569
4	2.0	4.53	13.0236	26.0473	4.00	169.6154	12.9514	0.0052	0.0123
5	2.5	5.34	13.1882	32.9704	6.25	173.9273	13.3173	0.0167	0.0029
6	3.0	9.50	13.7642	41.2927	9.00	189.4537	13.6832	0.0066	0.3966
7	3.5	13.50	14.1156	49.4047	12.25	199.2506	14.0490	0.0044	0.9628
8	4.0	18.50	14.4307	57.7228	16.00	208.2450	14.4149	0.0002	1.6804
Σ	18.0		105.075	244.102	51.00	1,385.888		$SS_E = 0.1710$	$S_{yy} = 5.7930$

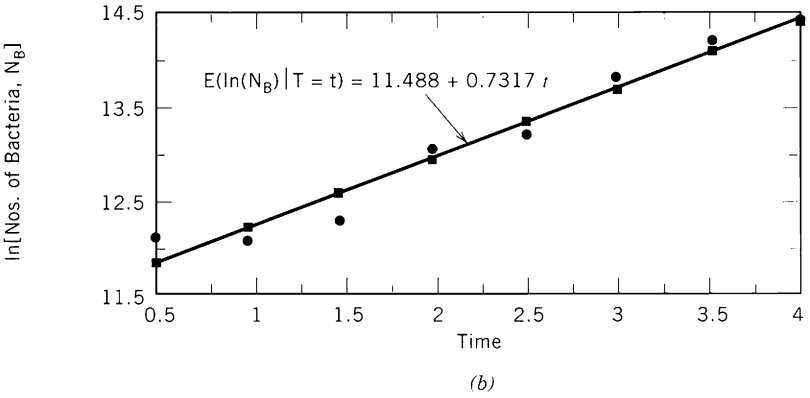


Figure 6.6b ln N_B Versus Time

General Comments

The analysis of scatter diagrams or residual analysis will indicate whether the relationship between the variables is linear or nonlinear. Determining the exact form of the nonlinear relationship is not simple in many cases, since numerous types of nonlinear terms can be considered, (e.g., square, cube or higher order terms, mixed terms, logarithmic transformation, etc.). Statistical methods like Box–Cox (1964) can be used for this purpose. These advanced topics are beyond the scope of this book.

EXAMPLE 6.17

As discussed in Section 6.6, the looseness or denseness of a sand deposit can be estimated directly by measuring the relative density, D_r, or indirectly using the SPT-value or N value. Since D_r and N measure the same soil characteristic, a relationship between them is expected. The following form of the relationship is suggested:

$$E(D_r \mid N = n) = b_0 + b_1\sqrt{n}. \tag{6.79}$$

To determine the regression coefficients b_0 and b_1, an investigation was carried out and the following results were obtained:

D_r	N
0.30	4
0.35	6
0.50	8
0.55	10
0.66	15
0.70	18
0.72	20
0.80	25
0.95	30

(a) Determine the relationship between D_r and N.

(b) Calculate the coefficient of determination of the regression equation.

SOLUTION

(a) In this case, the independent or regressor variable is $\sqrt{N}$. The scatter diagram for the problem is shown in Figure 6.7a. The information required to formulate the regression equation is summarized in Table 6.4. Equations 6.55 and 6.56a can be used to estimate the two regression coefficients:

$$\hat{b}_1 = \frac{22.5601 - \frac{1}{9} \times 33.5051 \times 5.53}{136 - \frac{1}{9} \times (33.5051)^2} = 0.1751$$

and

$$\hat{b}_0 = \frac{1}{9} \times 5.53 - \frac{0.1751}{9} \times 33.5051 = -0.0374.$$

Table 6.4 Regression Analysis on D_r and N-value

No.	n_i	d_{R_i}	$n'_i = \sqrt{n_i}$	$n'_i d_{R_i}$	n'^2_i	$d^2_{R_i}$	$\hat{d}_{R_i}$	$(d_{R_i} - \hat{d}_{R_i})^2$	$(d_{R_i} - \bar{d}_R)^2$
1	4	0.30	2.0000	0.6000	4	0.0900	0.31275	0.0002	0.0989
2	6	0.35	2.4495	0.8573	6	0.1225	0.39147	0.0017	0.0699
3	8	0.50	2.8284	1.4142	8	0.2500	0.45783	0.0018	0.0131
4	10	0.55	3.1623	1.7393	10	0.3025	0.51629	0.0011	0.0042
5	15	0.66	3.8730	2.5562	15	0.4356	0.64074	0.0004	0.0021
6	18	0.70	4.2426	2.9698	18	0.4900	0.70548	0.0000	0.0073
7	20	0.72	4.4721	3.2199	20	0.5184	0.74567	0.0007	0.0111
8	25	0.80	5.0000	4.0000	25	0.6400	0.83810	0.0015	0.0344
9	30	0.95	5.4772	5.2034	30	0.9025	0.92167	0.0008	0.1126
Σ		5.53	33.5051	22.5601	136	3.7515		$SS_E = 0.0082$	$S_{yy} = 0.3536$

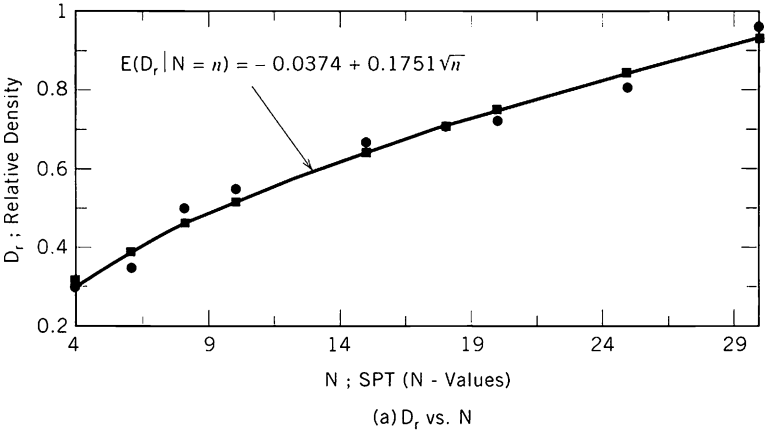


Figure 6.7a D_r Versus N

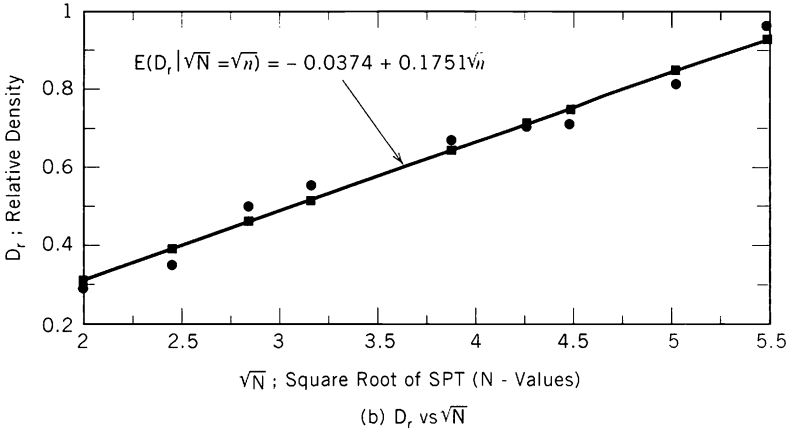


Figure 6.7b D_r Versus $\sqrt{N}$

Thus, the regression equation is

$$E(D_r \mid N = n) = -0.0374 + 0.1751\sqrt{n}.$$

The regression equation between D_r and $\sqrt{N}$ is shown in Figure 6.7b.

(b) Using the information given in Table 6.4, we can calculate the coefficient of determination of the regression equation as

$$R^2 = 1 - \frac{0.0082}{0.3536} = 0.977.$$

6.7 CONCLUDING REMARKS

In many practical engineering problems, the statistical information on a random variable cannot be obtained directly but needs to be evaluated indirectly from statistical information on other random variables. The concept of uncertainty evaluation of the response variable is introduced in this chapter. Some of the methods that can be used for this purpose are identified. The choice of method depends on whether the response variable is a known function (linear or nonlinear) of one or more random variables, or whether the functional relationship is known or unknown. Some of the methods are approximate and provide only limited statistical information on the response variable in terms of its mean and variance, without providing any information on the underlying distribution. In many cases this limited information is valuable if used intelligently.

When the functional relationship between variables is not known and at least one of the variables is random, the regression analysis technique can be used to develop a statistical relationship. Procedures were presented to formulate a linear or nonlinear relationship between two or more variables. The adequacy of the regression equation was evaluated by calculating the coefficient of determination. Various assumptions and limitations of regression analysis were also discussed.

6.8 PROBLEMS

6.1 The head loss due to sudden enlargement in a pipeline, as shown in Figure P6.1, can be calculated as

$$h_l = \frac{V_1^2}{2g}\left(1 - \frac{A_1}{A_2}\right)^2$$

where A_1 and V_1 are the area and velocity of the fluid in the smaller pipe, and A_2 and V_2 are the area and velocity of the fluid in the larger pipe. If A_1/A_2 is a constant of magnitude 0.5 and V_1 has an exponential distribution with mean of μ_{V_1}, determine the distribution of h_l.

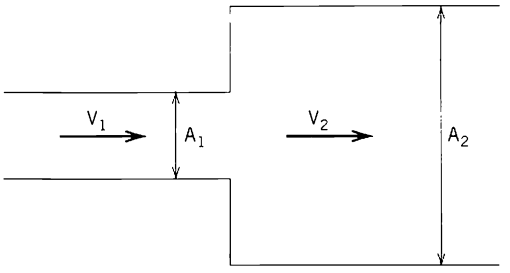


Figure P6.1 Sudden Expansion of Flow in a Pipe

6.2 In Problem 6.1, if V_1 is normally distributed with parameters μ_{V_1} and σ_{V_1}, determine the distribution of h_l.

6.3 The drag force, F_D, acting on an immersed body by a moving fluid can be calculated as

$$F_D = C_D A \frac{\rho U^2}{2}$$

where C_D is the drag coefficient, A is the projected area of the body on a plane normal to the flow, ρ is the mass density of the fluid, and U is the undisturbed velocity of the fluid. Suppose C_D, A, and ρ are known constants of values 0.6, 10 ft^2, and 1.94 slug/ft^3, respectively. U is a lognormal random variable with parameters λ_U and ζ_U.

Determine the distribution of F_D.

6.4 In Problem 6.3, if U has a normal distribution with parameters μ_U and σ_U, determine the distribution of F_D.

6.5 Consider a saturated clay layer of thickness H with an existing effective overburden pressure p_0 at the midheight of the layer. The construction of a building above it causes an increase in pressure, Δp. It is necessary to calculate the settlement S associated with the construction. Considering only the primary settlement, S can be calculated as

$$S = \frac{C_c H}{1 + e_0} \log\left(\frac{p_0 + \Delta p}{p_0}\right)$$

where C_c is the compression index, e_0 is the initial void ratio, and the other parameters were discussed earlier. Suppose H, p_0, Δp, and e_0 are known constants of value 4 m, 98 kN/m^2, 13 kN/m^2, and 1.0, respectively. For the given void ratio, the compression index is normally distributed with a mean of 0.27 and a standard deviation of 0.04.

(a) What is the distribution of the settlement S? Calculate the parameters of the distribution.

(b) If the allowable settlement is 40 mm, what is the probability that the building will suffer damage due to excessive settlement?

6.6 To calculate the bearing capacity of soil under a strip footing, Terzaghi suggested the following equation:

$$q_u = cN_c + qN_q + \frac{1}{2}\gamma BN_\gamma$$

where c is the cohesion, q is the surcharge, γ is the unit weight of the soil, B is the width of the foundation, and N_c, N_q, and N_γ are the bearing capacity factors. q can be calculated as γH, where H is the depth of the foundation. For a particular site, consider that $\gamma = 115$ pcf, $H = 4$ ft, $B = 4$ ft, $N_c = 17.69$, $N_q = 7.44$, and $N_\gamma = 3.64$, and assume that all these parameters are constants. c is a normally distributed random variable with a mean of 400 psf and a COV of 10%.

(a) What is the distribution of q_u and its mean and standard deviation?

(b) What is the probability that the ultimate bearing capacity is less than 9,000 psf?

(c) Suppose that the allowable bearing capacity q_a is defined as $q_u/3$. What is the probability that it is less than 3,000 psf?

6.7 Suppose that instead of a strip footing, a square footing needs to be designed for a column at the site considered in Problem 6.6. Terzaghi suggested the following equation for a square footing:

$$q_u = 1.3cN_c + \gamma HN_q + 0.4\gamma BN_\gamma.$$

Consider a square footing of size 4 ft × 4 ft.

(a) Determine the distribution of q_u and its mean and standard deviation.

(b) Using a safety factor of 3, determine the distribution of q_a and its mean and standard deviation.

(c) If an axial load of 60 kip is applied to the column, what is the probability of failure of the column?

6.8 A 15-ft deep-cut slope (i.e., $H = 15$ ft) has to be made in a soft, saturated clay deposit with its side rising at an angle of 70°, as shown in Figure P6.8. The average cohesion c_d that will be developed along the sliding surface in an undrained condition can be calculated as

$$c_d = m\gamma H$$

where γ is the unit weight of the soil deposit (assumed to be 110 pcf in this case), H is the depth of the cut of 15 ft, and m is a nondimensional parameter referred to as the stability number. The slope will fail if c_d is greater than the undrained shear strength c_u. Assume H and γ are known constants in this problem and c_u is 500 psf. The stability number m corresponding to the slope of 70° can be calculated as 0.215 (Das, 1998).

Assume m is a normally distributed normal random variable with a mean of 0.215 and a COV of 0.2.

(a) Determine the distribution of c_d and its mean and standard deviation.

(b) Calculate the probability of failure of the cut.

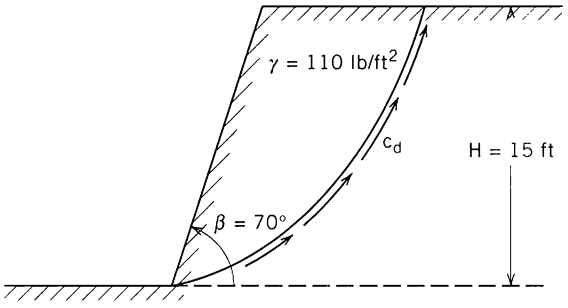


Figure P6.8 A deep-cut slope

6.9 The total gravity load on each floor of a five-story building consists of dead load and live load and can be expressed as

$$T = D + AX$$

where D is the dead load, A is the floor area, and X is the live load. D and A are assumed to be deterministic quantities of values 600 kip and 10,000 ft^2, but X is assumed to be a normal random variable with a mean of 0.015 kip/ft^2 and a COV of 0.40. Assume further that X in one floor is statistically independent of X in another floor.

(a) Suppose the design load L for each floor is defined as the 95th-percentile value of T; determine L.

(b) What is the probability that the design load L will be exceeded on more than one floor of the five-story building?

(c) Suppose the strength, S, of the soil supporting the five-story building is also a normal variable with a mean of 4,200 kip and a COV of 0.20. The structure will be seriously damaged if the total gravity load of the building exceeds the soil strength. What is the probability that the structure will be seriously damaged? Assume the soil strength is statistically independent of the gravity load.

6.10 To estimate the settlement due to consolidation of a homogeneous saturated soil deposit, the information on the coefficient of consolidation, c_v, is very important. It can be calculated as

$$c_v = \frac{k}{m_v \gamma_w}$$

where k is the hydraulic conductivity or coefficient of permeability, m_v is the coefficient of volume compressibility, and γ_w is the unit weight of water. For this example consider γ_w to have a constant value of 9.81 kN/m^3. Assume that k and m_v are statistically independent lognormal random variables with means of 1.3×10^{-7} m/min and 0.0011 m^2/kN, respectively. Both have COVs of 10%.

(a) Determine the distribution of c_v and the parameters of the distribution.

(b) What is the probability that c_v is greater than 1.2×10^{-5} m^2/min?

6.11 In Problem 6.8, suppose m and γ are statistically independent lognormal random variables with means of 0.215 and 110 pcf, respectively. The corresponding COVs are 0.2 and 0.05, respectively.

(a) Determine the distribution of c_d and the parameters of the distribution.

(b) What is the probability of failure of the cut?

6.12 Seismic activity in the San Francisco area may come from three major fault systems: the San Andreas (S), Calaveras (C), and Hayward (H) faults. Assume that earthquakes of magnitudes greater than 4 occur in these faults according to the independent Poission process, with mean activity rates of 0.6, 0.25, and 0.2 per year, respectively.

(a) What is the probability that there will be no earthquake with a magnitude greater than 4 in the San Francisco area next year?

(b) What is the probability that there will be at least one earthquake with a magnitude greater than 4 next year?

(c) What is the probability that there will be at least one earthquake with a magnitude greater than 4 in the next 2 years?

6.13 The standard penetration test value N and the relative density D_r are intended to measure the degree of compactness of in situ sand. For normally consolidated sand, Marcuson and Bieganousky (1977) suggested the following relationship:

$$D_r = 11.7 + 0.76 \times \left[1222N + 1600 - 53\sigma'_v - 50C_u^2 \right]^{1/2}$$

where σ'_v is the effective vertical stress in psi and C_u is the uniformity coefficient. Assume C_u is a constant of value 1.5, and the relative density needs to be estimated at a depth where the effective vertical stress is 10 psi. At that depth, N of 15 is measured. Assume that this is the mean value of N and has a COV of 0.25.

(a) Calculate the first-order mean and standard deviation of D_r.

(b) Suppose D_r is normally distributed with the mean and standard deviation obtained in Part (a). What is the probability that D_r is less than 55%?

6.14 The total shear resistance F of soil between B and C against slope failure, shown in Figure P6.14, can be calculated as

$$F = (C + P \tan \phi)L$$

where C is the cohesion, P is the pressure normal to the arc BC, ϕ is the friction angle of the soil, and L is the length of the arc BC. L is assumed to be 10 ft, a known constant. However, C, P, and ϕ are statistically independent random variables with mean values of 0.5 ksf, 2 ksf, and 25°, respectively, and corresponding COVs of 0.20, 0.10, and 0.20, respectively. No information on their distribution is available.

(a) Determine the first-order mean and variance of F.

(b) Determine the second-order mean of F.

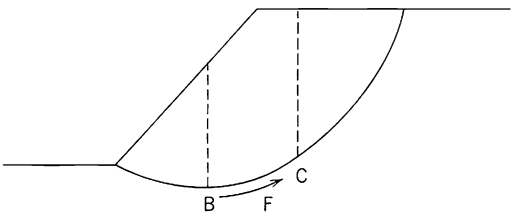


Figure P6.14 Shear Strength Evaluation

6.15 In Problem 6.3, suppose C_D, A, ρ, and U are statistically independent random variables with means of 0.5, 10 ft^2, 1.94 slug/ft^3, and10 ft/s, respectively, and corresponding COVs of 0.1, 0.05, 0.05, and 0.2, respectively. However, no information is available on their distribution.

(a) Determine the first-order mean and variance of F_D.

(b) Determine the second-order mean of F_D.

6.16 Consider Problem 6.15 again. Using the Taylor series finite difference procedure, calculate the first-order variance of F_D.

6.17 The wind pressure acting on a circular tank can be calculated from

$$p = 0.00256 K_z K_{zt} V^2 I G_f C_f$$

where K_z is the velocity pressure exposure coefficient evaluated at height z, K_{zt} is the topographical factor, V is the basic wind speed, I is the importance factor, G is the gust effect factor, and C_f is the force coefficient. For a given tank, the mean values of K_z, V, G_f, and C_f are 1.00, 0.7 × 90, 0.85, and 0.7, respectively. The corresponding uncertainties in terms

of COV are 0.1, 0.16, 0.12, and 0.1, respectively. Assume K_{zt} and I are constants of magnitude 1.0. Assume all the random variables are statistically independent.

(a) Determine the first-order mean and variance of p.

(b) Determine the second-order mean of p.

6.18 Using the Taylor series finite difference procedure, calculate the first-order variance of p in Problem 6.17.

6.19 To estimate the depth of penetration, X_0, of projectiles into concrete barriers, an extensive experimental investigation is carried out. Based on the experimental outcomes, a dimensionless impact factor I is introduced, which is a function of the weight, velocity, and diameter of a projectile and the compressive strength of the concrete barrier. The ratio of X_0 and the diameter of the projectile is denoted as Y. The relationship of Y and I is then studied. The following 10 sets of values for Y and I are available:

Y	I
0.70	3.0
0.78	3.5
0.80	4.0
0.92	5.0
1.15	8.0
1.20	10.0
1.50	13.0
1.65	15.0
1.95	19.0
2.00	20.0

(a) Develop a regression equation of the form $E(Y \mid i) = b_0 + b_1 i$.

(b) Calculate the conditional variance $\mathrm{Var}(Y \mid i)$.

(c) Calculate the coefficient of determination of the regression model.

6.20 To avoid using very irregular earthquake load time histories, equivalent uniform cyclic loadings N_{eq}, can be used, specifically in geotechnical engineering. Fifteen recorded earthquake time histories are considered and the corresponding N_{eq} values are calculated. The magnitude of the earthquake on the Richter scale M and the corresponding N_{eq} values are as follows:

M	N_{eq}
5.2	5.6
5.4	5.7
5.6	5.8
5.8	5.9
6.0	6.5
6.2	6.9
6.3	7.5
6.4	8.5
6.6	9.0
6.8	11.0
7.0	13.2
7.2	14.5
7.5	18.0
7.9	24.0
8.0	26.0

(a) Plot M along the X-axis and N_{eq} along the Y-axis.

(b) Assuming $\text{Var}(N_{eq} \mid m)$ is constant, develop a regression equation of the form $E(N_{eq} \mid m) = b_0 + b_1 m$.

(c) Calculate $\text{Var}(N_{eq} \mid m)$.

(d) Calculate the coefficient of determination.

6.21 Consider again the data on M and N_{eq} given in Problem 6.20.

(a) Assuming $\text{Var}(N_{eq} \mid m)$ is constant, develop a regression equation of the form $E(N_{eq} \mid m) = b_0 + b_1 m + b_2 m^2$.

(b) Calculate $\text{Var}(N_{eq} \mid m)$ and the coefficient of determination.

(c) Comment on the predictability of the two regression equations developed in Problems 6.20 and 6.21.

6.22 To develop a relationship between the peak shear stress τ and the normal stress σ', seven drained direct shear tests on an overconsolidated clay were conducted as follows:

σ' (kN/m^2)	τ (kN/m^2)
75.0	75.7
100.0	92.6
150.0	115.2
200.0	143.0
250.0	165.0
270.0	179.2
290.0	190.2

(a) Plot the data and develop a regression equation of the form $E(\tau \mid \sigma') = b_0 + b_1 \sigma'$.

(b) Calculate the cohesion of the clay deposit. Hint: It is the intercept b_0.

6.23 The monthly water consumption, C, in a city depends on the average daily temperature, T, and the rainfall, R, in a particular month. Records for a 12-month period are as follows:

C ($\times 10^6$ gallons)	T (°F)	R (inch)
45	60	3
40	55	I
55	60	0
60	65	1
60	70	2
80	75	2
90	80	1
80	80	3
70	75	2
65	70	3
60	65	2
60	60	4

(a) Assuming the conditional variance $\text{Var}(C \mid t,r)$ is constant, the following regression equation is suggested:

$$E(C \mid t,r) = b_0 + b_1 t + b_2 r.$$

Estimate the three regression coefficients and the conditional variance.

(b) Calculate the coefficient of determination of the regression equation.

6.24 The minimum safe stopping distance (S) for vehicles on level roads depends on the speed (v) at the beginning of the deceleration and the coefficient of friction (f) between the tires and the pavement. The following relationship is suggested to estimate the stopping distance for wet pavement:

$$E(S \mid v, f) = av^b f^c$$

Using the following 20 sets of data, estimate a, b, and c.

S (ft)	v (mph)	f
9	10	0.40
22	15	0.36
43	20	0.29
80	25	0.27
74	30	0.40
125	35	0.33
170	40	0.31
190	45	0.36
185	48	0.40
35	18	0.33
185	38	0.27
95	33	0.36
65	28	0.40
160	42	0.36
110	36	0.40
215	50	0.40
100	30	0.31
142	40	0.36
3	5	0.27
75	25	0.30

Chapter 7

Fundamentals of Reliability Analysis

7.1 INTRODUCTORY COMMENTS

In general, engineering design consists of proportioning the elements of a system so that it satisfies various criteria of performance, safety, serviceability, and durability under various demands. For example, a structure should be designed so that its strength or resistance is greater than the effects of the applied loads. However, there are numerous sources of uncertainty in the load- and resistance-related parameters. Methods for identifying and quantifying uncertainties in random variables were discussed in Chapters 5 and 6. Incorporating the information on uncertainty into actual design problems is the subject of this chapter.

7.2 DETERMINISTIC AND PROBABILISTIC APPROACHES

In the presence of uncertainty, it is not simple to satisfy the basic design requirements. Figure 7.1 shows a simple case considering two variables (one relating to the demand on the system, e.g., load on the structure, S, and the other to the capacity of the system, e.g., resistance of the structure, R). Both S and R are random in nature; their randomness is characterized by their means μ_S and μ_R, standard deviations σ_S and σ_R, and corresponding probability density functions $f_S(s)$ and $f_R(r)$, respectively, as shown in Figure 7.1. Figure 7.1 also identifies the deterministic (nominal) values of these parameters, S_N and R_N, used in a conventional safety factor-based approach. The purpose of this chapter is to develop a rational procedure to incorporate the information in Figure 7.1 into actual designs.

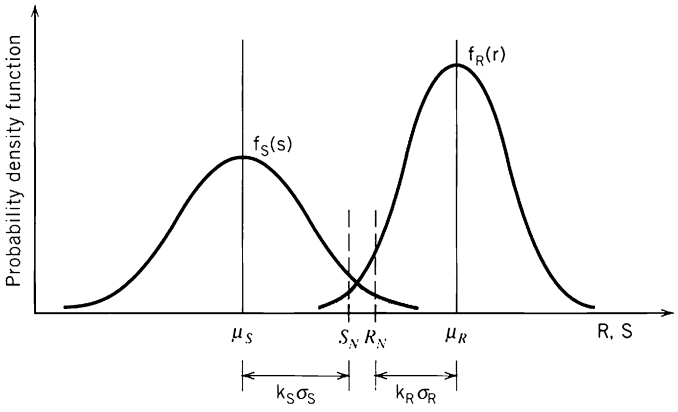


Figure 7.1
Fundamentals of Risk Evaluation

The concept of risk-based design was introduced by Freudenthal (1956) and was summarized by Freudenthal, Garrelts, and Shinozuka (1966). The concept has matured since then and is presented in the following sections.

7.3 RISK AND SAFETY FACTORS CONCEPT

Before the risk-based design format is developed, the concept of risk and the corresponding safety factor need some elaboration. The application of the risk-based format in various engineering disciplines can be described, at best, as nonuniform. Most of the progress has been made in structural engineering. Design guidelines using the load and resistance factor design (LRFD) concept are essentially based on the risk-based design format. In the following discussion, the risk-based design format in structural engineering is emphasized. However, the same concept can also be applied to other engineering disciplines by replacing resistance and load by supply and demand as discussed in Section 1.1 of Chapter 1. We hope the following discussion will help to accelerate the implementation of the risk-based design concept in other engineering disciplines, where necessary. Referring to Figure 7.1, design safety is ensured in a deterministic approach by requiring that R_N be greater than S_N with a specified margin of safety as

$$\text{Nominal } S_F = R_N / S_N \tag{7.1}$$

where S_F is the safety factor.

The nominal resistance (or capacity) R_N is usually a conservative value, perhaps one, two, or three standard deviations below the mean value. The nominal load (or demand) S_N is also a conservative value; however, it is several standard deviations above the mean value. Thus, the intended conservatism introduced in designs in the form of the nominal safety factor depends on many other factors, namely the uncertainty in the load and resistance and how conservatively the nominal load and resistance values are selected. The nominal safety factor may fail to convey the actual margin of safety in a design.

Conceptually, then, in a deterministic design the nominal safety factor can be applied to the resistance, to the load, or to both. The allowable stress design methods use a safety factor to compute the allowable stresses in members from the ultimate stress, and a successful design ensures that the stresses caused by the nominal values

of the loads do not exceed the allowable stresses. In other words, referring to Figure 7.1 and Equation 7.1, R_N is divided by a safety factor to compute the allowable resistance R_a, and safe design requires that the condition $S_N < R_a$ be satisfied. In this case, the safety factor is used for the resistance only. In the ultimate strength design method, the loads are multiplied by certain load factors to determine the ultimate loads, and the members are required to resist various design combinations of the ultimate loads. That is, in Figure 7.1, S_N is multiplied by a load factor to obtain the ultimate load S_u, and safe design requires the satisfaction of the condition $S_u < R_N$. In this case, the safety factors are used in the loads and in load combinations. In some designs, for example, in concrete design or in steel design using the load and resistance factor design (LRFD) concept, the capacity reduction factor (generally less than one) and load factors (generally more than one) are used to achieve the same objective. Essentially, the safety factors are used to estimate both the resistance and the loads.

The intent of these conventional approaches can be explained by considering the area of overlap between the two curves (the shaded region in Figure 7.1), which provides a *qualitative measure* of the probability of failure. This area of overlap depends on three factors:

1. The relative positions of the two curves: As the distance between the two curves increases, reducing the overlapped area, the probability of failure decreases. The positions of the curves may be represented by the means (μ_R and μ_S) of the two variables.
2. The dispersion of the two curves: If the two curves are narrow, then the area of overlap and the probability of failure are small. The dispersion may be characterized by the standard deviations (σ_R and σ_S) of the two variables.
3. The shapes of the two curves: The shapes are represented by the probability density functions $f_R(r)$ and $f_S(s)$.

The objective of safe design in deterministic design procedures can also be achieved, perhaps more comprehensively, by selecting the design variables in such a way that the area of overlap between the two curves is as small as possible, so that the underlying risk is not compromised within the constraints of economy. Conventional design approaches achieve this objective by shifting the positions of the curves through the use of safety factors. A more rational approach would be to compute the risk by accounting for all three overlap factors, and selecting the design variables so that an acceptable risk of failure is achieved. This is the foundation of the risk-based design concept. With this approach, however, the information on the probability density functions of the resistance and loads (as in Figure 7.1) is usually difficult to obtain, and engineers must formulate an acceptable design methodology using only the information on means and standard deviations.

7.4 RISK-BASED DESIGN CONCEPT AND THE DEVELOPMENT OF THE RISK-BASED DESIGN FORMAT

Instead of using the safety factor for the resistance alone, as in the working stress method, or for the loads alone, as in the ultimate strength method, it is more rational to apply safety factors to both resistance and loads, as is done in concrete or steel (LRFD)

structural design. The following idealized discussion is presented to develop a parallel between the deterministic approaches in terms of the capacity reduction factor and load factors with risk-based design. Referring to Figure 7.1, where the uncertainties in the load and resistance variables are expressed in the form of the probability density functions, we can express the measure of risk in terms of the probability of the *failure event* or $P(R < S)$ as:

$$\begin{aligned} p_f &= P(\text{failure}) = P(R < S) \\ &= \int_0^\infty \left[\int_0^s f_R(r)dr \right] f_S(s)ds \\ &= \int_0^\infty F_R(s) f_S(s)ds \end{aligned} \tag{7.2}$$

where $F_R(s)$ is the CDF of R evaluated at s. Equation 7.2 states that when the load is $S = s$, the probability of failure is $F_R(s)$, and since the load is a random variable, the integration needs to be carried out for all the possible values of S, with their respective likelihoods represented by the PDF of S. Equation 7.2 can be considered to be the basic equation of the risk-based design concept. The CDF of R or the PDF of S may not always be available in explicit form, and thus the integration of Equation 7.2 may not be practical. However, Equation 7.2 can be evaluated easily, without performing the integration, for some special cases. They are considered first in the following sections.

7.4.1 Load and Resistance Normal Variables: Single Load Case

Consider a structure with resistance R subjected to a single load S. The structure is subjected to one load at a time (i.e, dead load alone, live load alone, wind load alone, seismic load alone, etc.). If both R and S are normal variables, that is, $N(\mu_R, \sigma_R)$ and $N(\mu_S, \sigma_S)$, then another random variable Z can be introduced as

$$Z = R - S. \tag{7.3}$$

Since it is quite reasonable to assume that R and S are statistically independent, based on the discussion in Section 6.3.1.2, we can infer that Z is also a normal random variable, that is, $N(\mu_R - \mu_S, \sqrt{\sigma_R^2 + \sigma_S^2})$. Then, Equation 7.2 can be used to define the probability of failure as

$$p_f = P(Z < 0)$$

or

$$p_f = \Phi\left[\frac{0 - (\mu_R - \mu_S)}{\sqrt{\sigma_R^2 + \sigma_S^2}} \right]$$

or

$$p_f = 1 - \Phi\left[\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \right] \tag{7.4}$$

where Φ is the CDF of the standard normal variate. To develop the explicit expression for the risk-based design format, Equation 7.4 can be rewritten as

$$\mu_R \geq \mu_S + \Phi^{-1}(1 - p_f)\sqrt{\sigma_R^2 + \sigma_S^2} \tag{7.5}$$

where $\Phi^{-1}(1 - p_f)$ is the value of the standard normal variate at the probability level $(1 - p_f)$. Introducing $\beta = \Phi^{-1}(1 - p_f)$, and considering the equality condition, we can rewrite Equation 7.5 as

$$\mu_R = \mu_S + \beta\sqrt{\sigma_R^2 + \sigma_S^2}. \tag{7.6}$$

From Equation 7.4,

$$\beta = \Phi^{-1}(1 - p_f) = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}. \tag{7.7}$$

If β is large, p_f will be small, implying that the underlying risk is small.

To eliminate the square root sign to separate R and S in Equation 7.6, a parameter ε can be introduced as

$$\varepsilon = \frac{\sqrt{\sigma_R^2 + \sigma_S^2}}{\sigma_R + \sigma_S}. \tag{7.8}$$

ε can be considered to be approximately 0.75 in most cases. Substituting Equation 7.8 into Equation 7.7, one obtains

$$\beta = \frac{\mu_R - \mu_S}{\varepsilon(\sigma_R + \sigma_S)}. \tag{7.9a}$$

After the variables are separated, the equation becomes

$$\mu_R - \varepsilon\beta\sigma_R = \mu_S + \varepsilon\beta\sigma_S$$

or

$$(1 - \varepsilon\beta\delta_R)\mu_R = (1 + \varepsilon\beta\delta_S)\mu_S \tag{7.9b}$$

where

$$\delta_R = \frac{\sigma_R}{\mu_R} \quad \text{and} \quad \delta_S = \frac{\sigma_S}{\mu_S}.$$

Before considering the nominal safety factor, using Equation 7.9b, we can introduce the concept of the *central safety factor* by taking the ratio of the mean values of the load and resistance (refer to Figure 7.1):

$$\bar{\zeta} = \frac{\mu_R}{\mu_S} = \frac{1 + \varepsilon\beta\delta_S}{1 - \varepsilon\beta\delta_R}. \tag{7.10}$$

Referring to Equation 7.9b, we can express the capacity reduction factor corresponding to the central safety factor as

$$\bar{\phi} = 1 - \varepsilon\beta\delta_R. \tag{7.11a}$$

The corresponding load factor can be shown to be:

$$\bar{\gamma} = 1 + \varepsilon\beta\delta_S. \tag{7.11b}$$

To define the nominal safety factor, the nominal or characteristic values of the load and resistance need to be introduced as

$$R_N = \mu_R(1 - k_R\delta_R) \tag{7.12a}$$

$$S_N = \mu_S(1 + k_S\delta_S). \tag{7.12b}$$

Again, the nominal value of R is k_R standard deviations below the mean and the nominal value of S is k_S standard deviations above the mean, as shown in Figure 7.1. The conventional or nominal safety factor, as in Equation 7.1, becomes

$$\zeta = \frac{R_N}{S_N} = \frac{\mu_R}{\mu_S} \bullet \frac{(1 - k_R\delta_R)}{(1 + k_S\delta_S)} = \left(\frac{1 + \varepsilon\beta\delta_S}{1 + k_S\delta_S}\right)\left(\frac{1 - k_R\delta_R}{1 - \varepsilon\beta\delta_R}\right). \tag{7.13}$$

Thus, after the variables are separated, the nominal capacity reduction factor and the load factor, in terms of $\phi R_N \geq \gamma_{S_N}$, can be shown to be

$$\phi = \frac{1 - \varepsilon\beta\delta_R}{1 - k_R\delta_R} \tag{7.14a}$$

$$\gamma = \frac{1 + \varepsilon\beta\delta_S}{1 + k_S\delta_S}. \tag{7.14b}$$

It is clear from these two equations that the probability-based capacity reduction factor and load factor convey more information than the corresponding deterministic factors. Both of them depend on four factors: ε, β, δ_R or δ_S, and k_R or k_S. Equation 7.14a indicates that for normal engineering design when $(\varepsilon\beta)$ is expected to be greater than k_R, the capacity reduction factor ϕ will be less than one, which is commonly assumed in a deterministic design. It also indicates that the capacity reduction factor is a function of β, representing the acceptable risk for the structure being considered. If β is large, implying that the acceptable risk is small, the capacity reduction factor is expected to be small given that the other parameters remain the same. For small acceptable risk, more conservatism is introduced in estimating the resistance by using a low ϕ factor. The capacity reduction factor also depends on the uncertainty in the resistance, δ_R. It can be shown that if the uncertainty in R is large, the ϕ factor will be smaller. It depends on how conservatively the nominal resistance value, represented by the parameter k_R, is selected. If k_R is large, indicating that the nominal resistance was selected very conservatively, then the ϕ factor will approach unity, as expected. The capacity reduction factor is also a function of ε, representing the uncertainty in both the resistance and load variables.

The load factor for normal engineering design ($\varepsilon\beta > k_S$), represented by Equation 7.14b, is expected to be greater than one. As stated earlier, its value depends on four factors. The load factor depends on the uncertainty in the load under consideration. If different loads have different amounts of uncertainty, the load factors are expected to be different while all the other parameters remain the same. The uncertainty in the dead load is expected to be smaller than that in the live load. Thus, the load factor for the dead load will be smaller than that of the live load. If β is large, implying that the acceptable risk for the project is small, the corresponding load factor will also be large. If the design load is selected very conservatively, implying that k_S is large, then the load factor will approach unity, as expected. If ε is large, the load factor will also be large.

In deterministic design, the capacity reduction factor and load factors are determined subjectively based on judgment, intuition, and experience; in probabilistic designs, they can be estimated explicitly project by project considering the specific conditions, giving more control to the design engineers.

7.4.2 Load and Resistance Normal Variables: Multiple Load Case

In engineering design, components need to be designed to meet the maximum demand considering all possible loads that may act on them during their lifetime. Considering one load at a time may not be sufficient; the likely combinations of loads need to be considered. It is extremely unlikely that all possible loads will act simultaneously on a structure, but some of them will act together. For example, in a typical structural design, some of the common load combinations are dead plus live loads, dead plus live plus wind loads, dead plus live plus seismic loads, and dead plus wind loads. Thus, it is essential that multiple load effects be considered when estimating load and resistance factors.

To consider the effect of multiple loadings, S can be represented as $S = S_1 + S_2 + ... + S_n$. An obvious choice in this case is to combine the multiple load effects into one, as was discussed extensively in Chapter 6. Since S is a linear function of other random variables, its mean and variance can be estimated using Equations 6.32 and 6.33 or 6.34. Then, assuming S is a normal random variable and using the discussion in the previous section, we can estimate the resistance and load factors. The load factor will give a composite value considering the effect of loads in that particular combination; it will not give the individual load factors.

To estimate the individual load factors, the following procedure can be followed. For the multiple loads case, assuming $S_1, S_2, \ldots, S_n$ are statistically independent as in Equation 6.34, Equation 7.6 can be rewritten as:

$$\mu_R = (\mu_{S_1} + \mu_{S_2} + \ldots + \mu_{S_n}) + \beta\sqrt{\sigma_R^2 + \left(\sigma_{S_1}^2 + \sigma_{S_2}^2 + \ldots + \sigma_{S_n}^2\right)}.$$

Introducing the parameter ε, similar to Equation 7.8, results in

$$\begin{aligned}\mu_R = {} & (\mu_{S_1} + \mu_{S_2} + \ldots + \mu_{S_n}) \\ & + \varepsilon\beta\left(\sigma_R + \sqrt{\sigma_{S_1}^2 + \sigma_{S_2}^2 + \ldots + \sigma_{S_n}^2}\right).\end{aligned} \tag{7.15}$$

Again, to help separate the load variables, and using the information on the individual mean and standard deviation of the loads, we can eliminate the square root sign in Equation 7.15 can by introducing a parameter ε_{nn}, similar to ε in Equation 7.8, as

$$\varepsilon_{nn} = \frac{\sqrt{\sigma_{S_1}^2 + \sigma_{S_2}^2 + \ldots + \sigma_{S_n}^2}}{\sigma_{S_1} + \sigma_{S_2} + \ldots + \sigma_{S_n}}. \tag{7.16}$$

Equation 7.15 can be rewritten as

$$\mu_R = (\mu_{S_1} + \mu_{S_2} + \ldots + \mu_{S_n}) + \varepsilon\beta[\sigma_R + \varepsilon_{nn}(\sigma_{S_1} + \sigma_{S_2} + \ldots + \sigma_{S_n})]$$

or

$$\begin{aligned}(1 - \varepsilon\beta\delta_R)\mu_R = {} & (1 + \varepsilon\varepsilon_{nn}\beta\delta_{S_1})\mu_{S_1} \\ & + (1 + \varepsilon\varepsilon_{nn}\beta\delta_{S_2})\mu_{S_2} + \ldots + (1 + \varepsilon\varepsilon_{nn}\beta\delta_{S_n})\mu_{S_n}.\end{aligned} \tag{7.17}$$

The form of Equation 7.17 is identical to that of Equation 7.9b. By following the same logic as for the single load case, we can show that for the case of multiple loads, the central and nominal capacity reduction factors can still be evaluated from Equations 7.11a and 7.14a, respectively. However, the central and nominal load factors for the ith load, similar to Equations 7.11b and 7.14b, respectively, become

$$\bar{\gamma}_i = 1 + \varepsilon\varepsilon_{nn}\beta\delta_{S_i} \tag{7.18a}$$

and

$$\gamma_i = \frac{1 + \varepsilon\varepsilon_{nn}\beta\delta_{S_i}}{1 + k_{S_i}\delta_{S_i}}. \tag{7.18b}$$

Suppose a structural member is subjected to dead load (D) and live load (L) only. For design purposes, the following relationship needs to be satisfied:

$$\phi R_N \geq \gamma_D D + \gamma_L L. \tag{7.19}$$

If the safety margin is defined by Equation 7.3, then ϕ, γ_D, and γ_L can be estimated by Equations 7.14a and 7.18b, respectively.

EXAMPLE 7.1

A simply supported steel beam with a 30-foot span has been designed to carry a dead load of 70 psf and a live load of 100 psf, as shown in Figure 7.2. The beams are spaced 10 feet apart and are continuously laterally supported by the concrete slab. Using A36 steel and the American Institute of Steel Construction (AISC)'s *Manual of Load and Resistance Factor Design,* an engineer suggests a steel section of W14 × 61. Based on the preceding discussion, what will be the corresponding resistance and load factors?

To calculate these factors, some additional information is required. Consider that the nominal dead load and live load are both selected to be two standard deviations above the corresponding mean values, and the nominal resistance of the steel section is selected to be two standard deviations below the mean value. Further assume that the uncertainties in the dead load and live load in terms of COV are 0.13 and 0.37, respectively. The uncertainty in the resistance of the steel section, considering the uncertainties in material properties, fabrication, and modeling, is 0.13. These COVs are typical values reported in the literature. Consider first that all the variables are normal random variables.

$$\text{Nominal dead load} = D_N$$

$$= \mu_D + 2\sigma_D = \mu_D\left(1 + 2\frac{\sigma_D}{\mu_D}\right) = \mu_D(1 + 2\delta_D)$$

or

$$70 = \mu_D(1 + 2 \times 0.13)$$

or

$$\mu_D = 55.56 \text{ psf} \quad \text{and} \quad \sigma_D = 0.13 \times 55.56 = 7.22 \text{ psf}.$$

Similarly, the mean and standard deviation of the live load can be shown to be

$$\mu_L = 57.47 \text{ psf} \quad \text{and} \quad \sigma_L = 0.37 \times 57.47 = 21.26 \text{ psf}.$$

It is quite logical to assume that the dead and live loads are statistically independent. Denoting $S = D + L$, and using Equations 6.21 and 6.22, we can show the mean and standard deviation of S to be

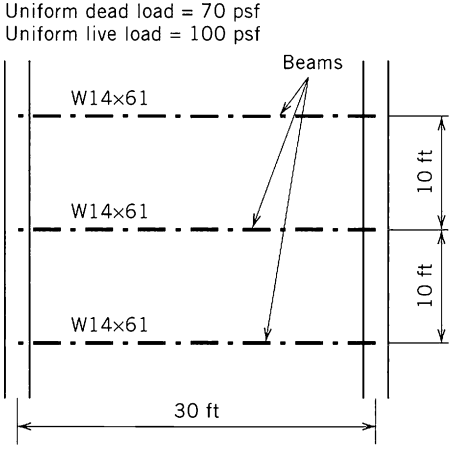


Figure 7.2 Resistance and Load Factor Evaluation for Beams

$$\mu_S = 55.56 + 57.47 = 113.03 \text{ psf}$$

and

$$\sigma_S = \sqrt{7.22^2 + 21.26^2} = 22.45 \text{ psf.}$$

Thus, $\delta_S = 22.45/113.03 = 0.199$.

For a simply supported beam of span l and subjected to a uniform load (per unit length) of S, the applied moment can be calculated using the formula $M = Sl^2/8$. Denoting the moment caused by the applied dead and live loads as M_A, we can estimate its mean and standard deviation as

$$\mu_{M_A} = (113.03 \times 10 \times 30^2 \times 12)/(8 \times 1{,}000) = 1{,}525.91 \text{ kip - in.}$$

and

$$\sigma_{M_A} = 0.199 \times 1{,}525.91 = 303.66 \text{ kip - in.}$$

$$\text{Nominal resistance} = R_N = \mu_R - 2\sigma_R.$$

The plastic moment capacity of a beam M_p can be considered to be the nominal moment capacity of the beam. Thus,

$$R_N = M_P = ZF_y = 102 \times 36 = 3{,}672 = \mu_R(1 - 2 \times 0.13)$$

or

$$\mu_R = 4{,}962.16 \text{ kip - in. and } \sigma_R = 0.13 \times 4{,}962.16 = 645.08 \text{ kip - in.}$$

Equation 7.4 can be used to estimate the probability of failure of the beam subjected to the dead and live loads considered here:

$$p_f = 1 - \Phi\left(\frac{4{,}962.16 - 1{,}525.91}{\sqrt{645.08^2 + 303.66^2}}\right)$$

$$= 1 - \Phi(4.82) = 0.72 \times 10^{-6}.$$

For this example β is 4.82. The resistance and load factors for this design can be estimated as discussed next. Using Equation 7.8, we can show that

$$\varepsilon = \frac{\sqrt{645.08^2 + 303.66^2}}{645.08 + 303.66} = 0.75.$$

Using Equation 7.16, we can calculate ε_{nn} as

$$\varepsilon_{nn} = \frac{\sqrt{7.22^2 + 21.26^2}}{7.22 + 21.26} = 0.79.$$

Using Equations 7.14a and 7.18b, we can estimate the resistance and load factors as

$$\phi = \frac{1 - 0.75 \times 4.82 \times 0.13}{1 - 2 \times 0.13} = 0.72$$

$$\gamma_D = \frac{1 + 0.75 \times 0.79 \times 4.82 \times 0.13}{1 + 2 \times 0.13} = 1.09$$

$$\gamma_L = \frac{1 + 0.75 \times 0.79 \times 4.82 \times 0.37}{1 + 2 \times 0.37} = 1.18.$$

Thus, the design equation is

$$0.72R = 1.09D + 1.18L$$

or

$$R = 1.51D + 1.64L.$$

7.4.3 Load and Resistance Lognormal Variables: Single Load Case

Considering the physical aspects of a design problem, R and S can be more appropriately considered to be statistically independent lognormal variables, that is, $LN(\lambda_R, \zeta_R)$ and $LN(\lambda_S, \zeta_S)$, since they cannot take negative values. In this case, another random variable Y can be introduced as

$$Y = R / S \tag{7.20a}$$

or

$$\ln Y = Z = \ln R - \ln S. \tag{7.20b}$$

The failure event can be defined as when $Y < 1.0$ or $Z < 0.0$. Since R and S are lognormal, $\ln R$ and $\ln S$ are normal (see Section 6.3.1.3); therefore, $\ln Y$ or Z is a normal random variable, that is, $Z \sim N\left(\lambda_R - \lambda_S, \sqrt{\zeta_R^2 + \zeta_S^2}\right)$. The probability of failure, similar to Equation 7.4, can be defined as

$$p_f = 1 - \Phi\left(\frac{\lambda_R - \lambda_S}{\sqrt{\zeta_R^2 + \zeta_S^2}}\right). \tag{7.21}$$

Using the relationships between the mean, standard deviation, and coefficient of variation and the parameters of the lognormal distribution (Equations 4.10 and 4.11), we can rewrite Equation 7.21 as

$$p_f = 1 - \Phi\left[\frac{\ln\left\{\left(\frac{\mu_R}{\mu_S}\right)\sqrt{\frac{1+\delta_S^2}{1+\delta_R^2}}\right\}}{\sqrt{\ln(1+\delta_R^2)(1+\delta_S^2)}}\right]. \tag{7.22}$$

If δ_R and δ_S are not large, say ≤ 0.30, Equation 7.22 can be simplified as

$$p_f \approx 1 - \Phi\left[\frac{\ln\left(\frac{\mu_R}{\mu_S}\right)}{\sqrt{\delta_R^2 + \delta_S^2}}\right]. \tag{7.23}$$

In this formulation, β as in Equation 7.7 can be shown to be

$$\beta = \Phi^{-1}(1 - p_f) = \frac{\ln\left\{\left(\frac{\mu_R}{\mu_S}\right)\sqrt{\frac{1+\delta_S^2}{1+\delta_R^2}}\right\}}{\sqrt{\ln(1+\delta_R^2)(1+\delta_S^2)}} \approx \frac{\ln\left(\frac{\mu_R}{\mu_S}\right)}{\sqrt{\delta_R^2 + \delta_S^2}}. \tag{7.24}$$

In many engineering problems of practical interest, the simplification suggested in Equation 7.23 may not be appropriate, since the uncertainty in many loads in terms of COV can be large (greater than 0.3). As in Equation 7.6, an alternative form of Equation 7.22 can be expressed as

$$\mu_R = \mu_S\left(\sqrt{\frac{1+\delta_R^2}{1+\delta_S^2}}\right)\exp\left[\beta\sqrt{\ln(1+\delta_R^2) + \ln(1+\delta_S^2)}\right] \tag{7.25}$$

where $\beta = \Phi^{-1}(1 - p_f)$, as given by Equation 7.24.

The parameter ε_l, similar to ε in Equation 7.8, can be introduced as:

$$\varepsilon_l = \frac{\sqrt{\ln(1+\delta_R^2)+\ln(1+\delta_S^2)}}{\sqrt{\ln(1+\delta_R^2)}+\sqrt{\ln(1+\delta_S^2)}}. \tag{7.26}$$

Proceeding as in the normal variables case in Section 7.4.1, the capacity reduction factor and the load factor corresponding to the central safety factor can be expressed as

$$\bar{\phi} = \frac{\exp\left[-\beta\varepsilon_l\sqrt{\ln(1+\delta_R^2)}\right]}{\sqrt{1+\delta_R^2}} \tag{7.27a}$$

and

$$\bar{\gamma} = \frac{\exp\left[\beta\varepsilon_l\sqrt{\ln(1+\delta_S^2)}\right]}{\sqrt{1+\delta_S^2}}. \tag{7.27b}$$

To obtain these factors with respect to the nominal safety factor, as in Equation 7.12, the nominal values of the resistance can be shown to be

$$\ln R_N = \ln \mu_R - k_R\delta_R$$

or

$$R_N = \mu_R \exp(-k_R\delta_R). \tag{7.28a}$$

Similarly, the nominal load can be expressed as

$$S_N = \mu_S \exp(k_S\delta_S). \tag{7.28b}$$

The nominal safety factor becomes

$$\zeta = \frac{R_N}{S_N} = \frac{\mu_R}{\mu_S}\frac{\exp(-k_R\delta_R)}{\exp(k_S\delta_S)}. \tag{7.29}$$

The corresponding nominal capacity reduction factor and load factor are

$$\phi = \bar{\phi}\exp(k_R\delta_R) \tag{7.30a}$$

and

$$\gamma = \bar{\gamma}\exp(-k_S\delta_S) \tag{7.30b}$$

$\bar{\phi}$ and $\bar{\gamma}$ can be estimated from Equations 7.27a and 7.27b, respectively.

7.4.4 Load and Resistance Lognormal Variables: Multiple Load Case

To consider the effect of statistically independent multiple load cases (i.e., when $S = S_1 + S_2 + \ldots + S_n$), the parameter ε_{nl}, similar to Equation 7.16, can be estimated by trial and error from the following equation as

$$\mu_S\left(\frac{\exp\left[\beta\varepsilon_l\sqrt{\ln(1+\delta_S^2)}\right]}{\sqrt{1+\delta_S^2}}\right) = \sum_{i=1}^{n}\mu_{S_i}\left(\frac{\exp\left[\beta\varepsilon_l\varepsilon_{nl}\sqrt{\ln(1+\delta_{S_i}^2)}\right]}{\sqrt{1+\delta_{S_i}^2}}\right). \tag{7.31}$$

The nominal capacity reduction factor can still be calculated by using Equation 7.30a; however, the load factor corresponding to the central safety factor for the ith load becomes

$$\bar{\gamma}_{S_i} = \frac{\exp\left[\beta \varepsilon_l \varepsilon_{nl} \sqrt{\ln(1+\delta_{S_i}^2)}\right]}{\sqrt{1+\delta_{S_i}^2}} \tag{7.32}$$

and the corresponding nominal load factor for the ith load is

$$\gamma_{S_i} = \bar{\gamma}_{S_i} \exp(-k_{S_i} \delta_{S_i}). \tag{7.33}$$

EXAMPLE 7.2

Example 7.1 is considered again, except that now both the load and resistance are assumed to be lognormal random variables. The means, standard deviations, and coefficients of variation of all the parameters were calculated earlier. In this case, the probability of failure of a beam can be calculated as

$$p_f = 1 - \Phi\left[\frac{\ln\left\{\left(\frac{4,962.16}{1,525.91}\right)\sqrt{\frac{(1+0.199^2)}{(1+0.13^2)}}\right\}}{\sqrt{\ln(1+0.13^2)(1+0.199^2)}}\right] = 1 - \Phi(5.05) = 0.2213 \times 10^{-6}.$$

In this case, β is 5.05. With Equation 7.26, ε_l can be calculated as

$$\varepsilon_l = \frac{\sqrt{\ln(1+0.13^2) + \ln(1+0.199^2)}}{\sqrt{\ln(1+0.13^2)} + \sqrt{\ln(1+0.199^2)}} = 0.72.$$

With Equation 7.31, ε_{nl} can be calculated by trial and error as:

$$113.03\left(\frac{\exp\left[5.05 \times 0.72 \times \sqrt{\ln(1+0.199^2)}\right]}{\sqrt{1+0.199^2}}\right)$$

$$= 55.56\left(\frac{\exp\left[5.05 \times 0.72 \times \varepsilon_{nl}\sqrt{\ln(1+0.13^2)}\right]}{\sqrt{1+0.13^2}}\right)$$

$$+ 57.47\left(\frac{\exp\left[5.05 \times 0.72 \times \varepsilon_{nl}\sqrt{\ln(1+0.37^2)}\right]}{\sqrt{1+0.37^2}}\right)$$

ε_{nl} is found to be 0.77. Equations 7.27a, 7.30a, 7.32, and 7.33 can be used to obtain the following information:

$$\bar{\phi} = \frac{\exp\left[-5.05 \times 0.72 \times \sqrt{\ln(1+0.13^2)}\right]}{\sqrt{1+0.13^2}} = 0.62$$

$$\bar{\gamma}_D = \frac{\exp\left[5.05 \times 0.72 \times 0.77 \times \sqrt{\ln(1+0.13^2)}\right]}{\sqrt{1+0.13^2}} = 1.42$$

$$\bar{\gamma}_L = \frac{\exp\left[5.05 \times 0.72 \times 0.77 \times \sqrt{\ln(1+0.37^2)}\right]}{\sqrt{1+0.37^2}} = 2.56$$

$$\phi = 0.62 \exp(2 \times 0.13) = 0.80$$

$$\gamma_D = 1.42 \exp(-2 \times 0.13) = 1.09$$

$$\gamma_L = 2.56 \exp(-2 \times 0.37) = 1.22$$

Thus, the design equation is

$$0.8R = 1.09D + 1.22L$$

or

$$R = 1.36D + 1.53L.$$

The two examples given here clearly indicate that it is not difficult to estimate the underlying resistance and load factors in a particular design for a set of assumptions. Whether these factors are acceptable is a different question. In actual building design codes, these factors are usually calibrated to satisfy current practice. Furthermore, assuming that the acceptable risk is going to remain the same for all the load combinations to be considered for the design of a structure, the engineer can calculate the corresponding resistance and load factors similarly for all load combinations. Maintaining uniform risk for different loads and load combinations is not practical in deterministic designs. The ability to design a structure for uniform risk with several loads and load combinations is one of the many desirable features of risk-based design.

The discussion and the examples clearly indicate that conventional safety factor–based deterministic designs, in terms of capacity reduction factor and load factors, and probability-based load and resistance factor designs are essentially parallel to each other. However, probabilistic design addresses the necessary design conservatism more explicitly, perhaps better and more comprehensively, through treatment of the uncertainty in the random variables, the conservatism used in selecting the design values, and the desired underlying reliability. Engineers are empowered to use judgment in selecting these factors on a case-by-case basis. This concept is the basis of all the reliability-based design codes being developed in different areas of engineering worldwide.

7.5 FUNDAMENTAL CONCEPT OF RELIABILITY ANALYSIS

The basic concept of the classical theory of structural reliability and risk-based design can now be presented more formally. We have seen that it is not difficult to calculate the underlying resistance and load factors for a given design, assuming an acceptable level of risk. However, it is more relevant to calculate the underlying risk of a given design, as is discussed in the following sections.

The first step in evaluating the reliability or probability of failure of a structure is to decide on specific performance criteria and the relevant load and resistance parameters,

called the basic variables X_i, and the functional relationships among them corresponding to each performance criterion. Mathematically, this relationship or *performance function* can be described as

$$Z = g(X_1, X_2, \ldots, X_n). \tag{7.34}$$

The *failure surface* or the *limit state* of interest can then be defined as $Z = 0$. This is the boundary between the safe and unsafe regions in the design parameter space, and it also represents a state beyond which a structure can no longer fulfill the function for which it was designed. Assuming R and S are the two basic random variables, the failure surface and the safe and unsafe regions are shown in Figure 7.3. The *limit state equation* plays an important role in the development of structural reliability analysis methods. A limit state can be an explicit or implicit function of the basic random variables, and it can be in simple or complicated form. Reliability analysis methods have been developed corresponding to limit states of different types and complexity, as discussed in the following sections.

Using Equation 7.34, we find that failure occurs when $Z < 0$. Therefore, the probability of failure, p_f, is given by the integral

$$p_f = \int \ldots \int_{g()<0} f_X(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n \tag{7.35}$$

in which $f_X(x_1, x_2, \ldots, x_n)$ is the joint probability density function for the basic random variables $X_1, X_2, \ldots, X_n$ and the integration is performed over the failure region, that is, $g(\,) < 0$. If the random variables are statistically independent, then the joint probability density function may be replaced by the product of the individual probability density functions in the integral.

Equation 7.35 is a more general representation of Equation 7.2. The computation of p_f by Equation 7.35 is called the *full distributional approach* and can be considered to be the fundamental equation of reliability analysis. In general, the joint probability density function of random variables is practically impossible to obtain. Even if this information is available, evaluating the multiple integral is difficult. Therefore, one approach is to use analytical approximations of this integral that are simpler to compute. To clarify the presentation, these methods can be grouped into two types, namely, *first-order reliability methods* (FORM) and *second-order reliability methods* (SORM).

The limit state of interest can be linear or nonlinear functions of the basic variables. FORM can be used to evaluate Equation 7.35 when the limit state function is a linear function of uncorrelated normal variables or when the nonlinear limit state function is repre-

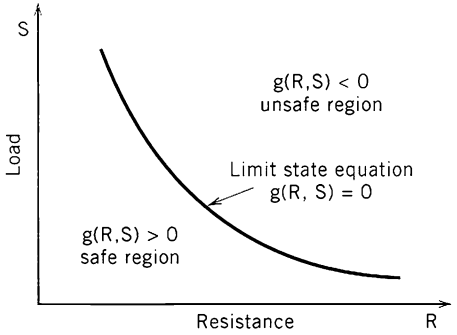


Figure 7.3 Limit State Concept

sented by a first-order (linear) approximation with equivalent normal variables, as is elaborated further in Section 7.6. SORM estimates the probability of failure by approximating the nonlinear limit state function (including a linear limit state function with correlated nonnormal variables) by a second-order representation. SORM is discussed in Chapter 8.

7.6 FIRST-ORDER RELIABILITY METHODS (FORM)

The development of FORM can be traced historically to second-moment methods, which used the information on first and second moments of the random variables. These are *first-order second-moment* (FOSM) and *advanced first-order second-moment* (AFOSM) methods. In FOSM methods, the information on the distribution of random variables is ignored; however, in AFOSM methods, the distributional information is appropriately used.

7.6.1 First-Order Second-Moment Method (FOSM) or MVFOSM Method

The FOSM method is also referred to as the *mean value first-order second-moment* (*MVFOSM*) method in the literature. The MVFOSM method derives its name from the fact that it is based on a first-order Taylor series approximation of the performance function linearized at the mean values of the random variables, and because it uses only second-moment statistics (means and covariances) of the random variables. The original formulation by Cornell (1969) uses the simple two-variable approach of the previous sections. A performance function in this case can be defined by Equation 7.3. Assuming that R and S are statistically independent normally distributed random variables, the variable Z is also normally distributed. As demonstrated earlier, its mean and variance can be readily determined. The event of failure is $R < S$, or $Z < 0$. The probability of failure is given by Equation 7.4. The probability of failure depends on the ratio of the mean value of Z to its standard deviation. This ratio is commonly known as the *safety index* or *reliability index* and is denoted as β:

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{7.36}$$

The probability of failure in terms of the safety index can be obtained by rewriting Equation 7.4 as

$$p_f = \Phi(-\beta) = 1 - \Phi(\beta). \tag{7.37}$$

An alternative formulation proposed by Rosenbleuth and Esteva (1972) may also be used, assuming that the variables R and S are statistically independent lognormal random variables. As discussed in Section 7.4.3, for physical reasons these variables are restricted to positive values; hence it is more reasonable to assume that they are lognormally distributed. The performance function in this case can be defined by Equation 7.20a or 7.20b. Z is again a normal random variable, and the probability of failure may be computed using Equation 7.21.

These formulations may be generalized for many random variables, denoted by a vector $\mathbf{X}$. Let the performance function be written as

$$Z = g(\mathbf{X}) = g(X_1, X_2, \ldots, X_n) \tag{7.38}$$

A Taylor series expansion of the performance function about the mean value gives

$$Z = g(\mu_{\mathbf{X}}) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}(X_i - \mu_{X_i}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 g}{\partial X_i \partial X_j}(X_i - \mu_{X_i})(X_j - \mu_{X_j}) + \ldots \tag{7.39}$$

where the derivatives are evaluated at the mean values of the random variables (X_1, X_2, . . ., X_n), and μ_{X_i} is the mean value of X_i. As discussed in Section 6.4.2, truncating the series at the linear terms, we obtain the first-order approximate mean and variance of Z as

$$\mu_Z \approx g\left(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}\right) \tag{7.40}$$

and

$$\sigma_Z^2 \approx \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial g}{\partial X_i}\frac{\partial g}{\partial X_j}\mathrm{Cov}(X_i, X_j) \tag{7.41}$$

where $\mathrm{Cov}(X_i, X_j)$ is the covariance of X_i and X_j.

If the variables are uncorrelated, then the variance is simply

$$\sigma_Z^2 \approx \sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)^2 \mathrm{Var}(X_i). \tag{7.42}$$

The safety index can be calculated by taking the ratio of the mean and standard deviation of Z as in Equation 7.36. It is important to repeat here that the performance function is linearized at the mean values of the random variables, reflecting the concept behind the MVFOSM method.

Using the safety index β, we can find the exact probability of failure in only a few cases. For example, if all the X_i's are statistically independent normal variables and if Z is a linear function of the X_i values, then Z is normal and the probability of failure is given by Equation 7.37. Similarly, if all the X_i's are statistically independent lognormal variables and if $g(\mathbf{X})$ is a multiplicative function of the X_i's, then $Z = \ln g(\mathbf{X})$ is normal and the probability of failure is given by Equation 7.37. However, in most cases it is not likely that all the variables are statistically independent normals or lognormals. Nor is it likely that the performance function is a simple additive or multiplicative function of these variables. In such cases, the safety index cannot be directly related to the probability of failure; nevertheless, it does provide a rough idea of the level of risk or reliability in the design. As shown in detail in Section 7.4, the MVFOSM approach could be used to derive a set of safety factors for loads and resistance, thereby establishing the consideration of design uncertainty on a logically sounder basis. The MVFOSM method was used to derive earlier versions of the reliability-based design formats, such as the American Institute of Steel Construction, Inc. (AISC, 1986), Canadian Standard Associations (CSA, 1974), and Comité European du Béton (CEB, 1976), to cite just a few examples.

However, the MVFOSM approach has some deficiencies. The method does not use the distribution information about the variables when it is available. The function $g(\,)$ in Equation 7.34 is linearized at the mean values of the X_i variables. When $g(\,)$ is nonlinear, significant error may be introduced by neglecting higher order terms. More importantly, the safety index defined by Equation 7.36 fails to be constant under different but mechanically equivalent formulations of the same performance function. For example, the safety margins defined as $(R - S < 0)$ in Equation 7.3 and $(R/S < 1)$ in

Equation 7.20a are mechanically equivalent. Yet the probabilities of failure given by Equations 7.4 and 7.21 or 7.22 are different for the two formulations. Furthermore, an engineering problem can be formulated in terms of stress or strength, as elaborated with the help of the following examples, and should produce identical results in either case. But the simplified method just discussed will give two different safety indexes. These observations can be explained with the help of simple examples.

EXAMPLE 7.3

A W16 × 31 steel section made of A36 steel is suggested to carry an applied deterministic bending moment of 1,140 kip-in. The nominal yield stress F_y of the steel is 36 ksi, and the nominal plastic modulus of the section Z is 54 in.^{3}. Consider that the distributions of these random variables are unknown; only the means, standard deviations, and COVs are known:

$$\mu_{F_y} = 38 \text{ ksi}, \ \sigma_{F_y} = 3.8 \text{ ksi}, \text{ and } \delta_{F_y} = 0.1$$

$$\mu_Z = 54 \text{ in.}^3, \ \sigma_Z = 2.7 \text{ in.}^3, \text{ and } \delta_Z = 0.05.$$

It is quite logical to assume that F_y and Z are statistically independent.

Strength Formulation

Considering the strength formulation first, the resistance $R = F_y Z$ and the load S = 1,140 kip-in. In this example, the load is a constant; thus μ_S = 1,140, σ_S = 0, and δ_S = 0. Using Equations 7.40 and 7.42, we can calculate the first-order mean and standard deviation of R as

$$\mu_R \approx \mu_{F_y} \mu_Z = 38 \times 54 = 2{,}052 \text{ kip-in.}$$

and

$$\sigma_R \approx \left[\text{Var}(F_y) \left(\frac{\partial R}{\partial F_y} \right)^2 + \text{Var}(Z) \left(\frac{\partial R}{\partial Z} \right)^2 \right]^{1/2} = \left[3.8^2 \times \mu_Z^2 + 2.7^2 \times \mu_{F_y}^2 \right]^{1/2}$$

$$= \left[(3.8 \times 54)^2 + (2.7 \times 38)^2 \right]^{1/2} = 229.42 \text{ kip-in.}$$

Thus, δ_R = 229.42/2,052 = 0.112.

Assuming the performance function to be of the form represented by Equation 7.3, we can write the limit state equation as

$$g() = F_y Z - 1{,}140 = 0. \tag{7.43}$$

The corresponding safety index, as in Equation 7.7, is

$$\beta = \frac{2{,}052 - 1{,}140}{\sqrt{(229.42)^2 + 0^2}} = 3.975.$$

If the performance function is assumed to be of the form represented by Equation 7.20, the corresponding safety index, according to Equation 7.24, becomes

$$\beta = \frac{\ln(2{,}052 / 1{,}140)}{\sqrt{(0.112)^2 + 0^2}} = 5.248.$$

Obviously, these two safety indexes and the corresponding probabilities of failure are quite different.

Stress Formulation

The same problem can also be formulated in term of stresses. The limit state equation in this case can be expressed as

$$g() = F_y - \frac{1{,}140}{Z} = 0. \tag{7.44}$$

In this case, the resistance R is represented by the random variable F_y, and the load $S = 1{,}140/Z$. Thus,

$$\mu_R = \mu_{F_y} = 38 \text{ ksi}$$

and

$$\sigma_R = \sigma_{F_y} = 3.8 \text{ ksi}$$

and $\delta_R = 0.1$.

The first-order mean and standard deviation of S can be calculated as

$$\mu_S \approx \frac{1{,}140}{\mu_Z} = \frac{1{,}140}{54} = 21.11 \text{ ksi}$$

$$\sigma_S \approx \left[Var(Z)\left(-\frac{1{,}140}{\mu_Z^2} \right)^2 \right]^{1/2} = \sigma_Z \frac{1{,}140}{\mu_Z^2} = 2.7\left(\frac{1{,}140}{54^2} \right) = 1.056 \text{ ksi}.$$

Thus

$$\delta_S = 1.056 / 21.11 = 0.05.$$

The safety index according to Equation 7.7 for the stress formulation is found to be

$$\beta = \frac{38 - 21.11}{\sqrt{(3.8)^2 + (1.056)^2}} = 4.282.$$

The corresponding safety index according to Equation 7.24 is

$$\beta = \frac{\ln(38 / 21.11)}{\sqrt{(0.1)^2 + (0.05)^2}} = 5.258.$$

The observations are summarized in Table 7.1.

Table 7.1 Variance Problem in the MVFOSM Method

	Normal	Lognormal
Strength formulation	3.975	5.248
Stress formulation	4.282	5.258

The results clearly indicate that the safety indexes depend on the formulation of the limit state equation as well as the underlying assumption about the distribution of the limit state.

In the early 1970s, this lack of invariance problem was observed by many researchers. It was overcome by the *advanced first-order second moment* (AFOSM) method proposed by Hasofer and Lind (1974) for normal variables, as discussed next.

7.6.2 AFOSM Method for Normal Variables (Hasofer–Lind Method)

The *Hasofer–Lind* (H–L) *method* is applicable for normal random variables. It first defines the reduced variables as

$$X_i' = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \quad (i = 1, 2, \ldots, n) \tag{7.45}$$

where X_i' is a random variable with zero mean and unit standard deviation. Equation 7.45 is used to transform the original limit state $g(\mathbf{X}) = 0$ to the reduced limit state, $g(\mathbf{X}') = 0$. The $\mathbf{X}$ coordinate system is referred to as the *original coordinate system*. The $\mathbf{X}'$ coordinate system is referred to as the *transformed* or *reduced coordinate system*. Note that if X_i is normal, X_i' is standard normal. These notations will be used throughout this chapter to denote different coordinate systems. The safety index β_{HL} is defined as the minimum distance from the origin of the axes in the reduced coordinate system to the limit state surface (failure surface). It can be expressed as

$$\beta_{HL} = \sqrt{(\mathbf{x}'^*)^t (\mathbf{x}'^*)}. \tag{7.46}$$

The minimum distance point on the limit state surface is called the *design point* or *checking point*. It is denoted by vector $\mathbf{x}^*$ in the original coordinate system and by vector $\mathbf{x}'^*$ in the reduced coordinate system. These vectors represent the values of all the random variables, that is, $X_1, X_2, \ldots, X_n$ at the design point corresponding to the coordinate system being used.

This method can be explained with the help of Figure 7.4. Consider the linear limit state equation in two variables,

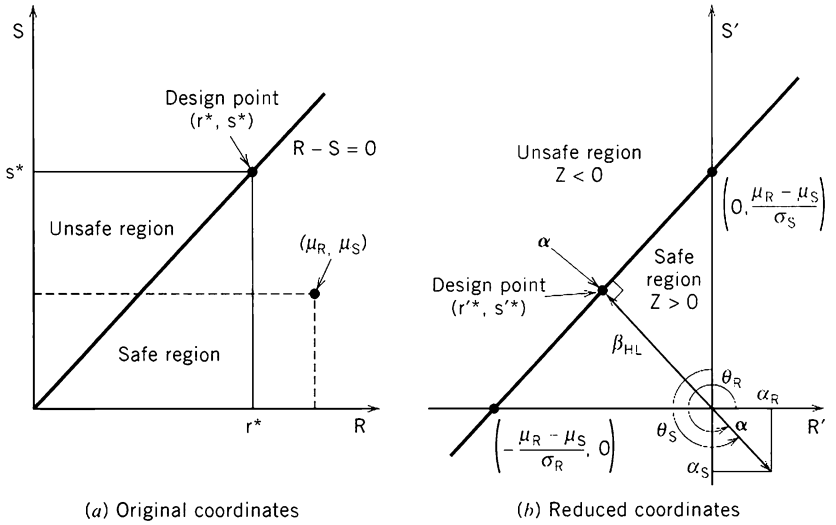
$$Z = R - S = 0. \tag{7.47}$$

This equation is similar to Equation 7.3. Note that R and S need not be normal variables. A set of reduced variables is introduced as

$$R' = \frac{R - \mu_R}{\sigma_R} \tag{7.48}$$

and

$$S' = \frac{S - \mu_S}{\sigma_S}. \tag{7.49}$$



(*a*) Original coordinates (*b*) Reduced coordinates

Figure 7.4 Hasofer–Lind Reliability Index: Linear Performance Function

If we substitute these into Equation 7.47, the limit state equation in the reduced coordinate system becomes

$$g(\ ) = \sigma_R R' - \sigma_S S' + \mu_R - \mu_S = 0. \tag{7.50}$$

The transformation of the limit state equation from the original to the reduced coordinate system is shown in Figure 7.4b. The safe and failure regions are also shown. From Figure 7.4b it is apparent that if the failure line (limit state line) is closer to the origin in the reduced coordinate system, the failure region is larger, and if it is farther away from the origin, the failure region is smaller. Thus, the position of the limit state surface relative to the origin in the reduced coordinate system is a measure of the reliability of the system. The coordinates of the intercepts of Equation 7.50 on the R' and S' axes can be shown to be $\left[-(\mu_R - \mu_S)/\sigma_R,\ 0\right]$ and $\left[0,\ (\mu_R - \mu_S)/\sigma_S\right]$, respectively. Using simple trigonometry, we can calculate the distance of the limit state line (Equation 7.50) from the origin as

$$\beta_{HL} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}. \tag{7.51}$$

This distance is referred to as the *reliability index* or *safety index*. It is the same as the reliability index defined by the MVFOSM method in Equation 7.36 if both R and S are normal variables. However, it is obtained in a completely different way based on geometry. It indicates that if the limit state is linear and if the random variables R and S are normal, both methods will give an identical reliability or safety index. This may not be true for other cases, as will be discussed further later.

In general, for many random variables represented by the vector $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ in the original coordinated system and $\mathbf{X}' = (X_1', X_2', \ldots, X_n')$ in the reduced coordinate system, the limit state $g(\mathbf{X}') = 0$ is a nonlinear function as shown in the reduced coordinates for two variables in Figure 7.5. At this stage, X_i''s are assumed to be uncorrelated. Consideration of correlated random variables is discussed in Chapter 8. Here, $g(\mathbf{X}') > 0$ denotes the safe state and $g(\mathbf{X}') < 0$ denotes the failure state. Again, the Hasofer–Lind reliability index β_{HL} is defined as the minimum distance from the origin to the design point on the limit state in the reduced coordinates and can be expressed by Equation 7.46, where $\mathbf{x}'^*$ represents the coordinates of the design point or the point of minimum distance from the origin to the limit state. In this definition the reliability index is invariant, because regardless of the form in which the limit state equation is written, its geometric shape and the distance from the origin remain constant. For the limit state surface where the failure region is away from the origin,

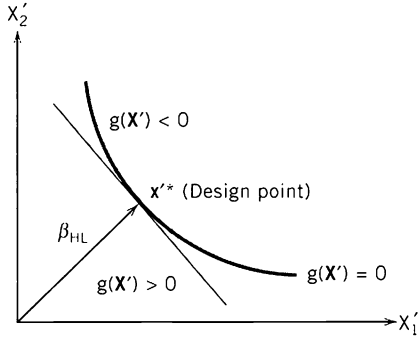


Figure 7.5 Hasofer–Lind Reliability Index: Nonlinear Performance Function

it is easy to see from Figure 7.5 that $\mathbf{x}'^*$ is the most probable failure point. As will be elaborated with the help of an example later, the Hasofer–Lind reliability index can be used to calculate a first-order approximation of the failure probability as $p_f = \Phi(-\beta_{HL})$. This is the integral of the standard normal density function along the ray joining the origin and $\mathbf{x}'^*$. It is obvious that the nearer $\mathbf{x}'^*$ is to the origin, the larger is the failure probability. Thus the minimum distance point on the limit state surface is also the most probable failure point. The point of minimum distance from the origin to the limit state surface, $\mathbf{x}'^*$, represents the worst combination of the stochastic variables and is appropriately named the *design point* or the *most probable point* (MPP) of failure.

For nonlinear limit states, the computation of the minimum distance becomes an optimization problem:

$$\begin{gathered}\text{Minimize } D = \sqrt{\mathbf{x}'^t \mathbf{x}'} \\ \text{Subject to the constraint } g(\mathbf{x}) = g(\mathbf{x}') = 0\end{gathered} \tag{7.52}$$

where $\mathbf{x}'$ represents the coordinates of the checking point on the limit state equation in the reduced coordinates to be estimated. Using the method of Lagrange multipliers, we can obtain the minimum distance as

$$\beta_{HL} = -\frac{\sum_{i=1}^{n} x_i'^* \left(\frac{\partial g}{\partial X_i'}\right)^*}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i'}\right)^{2^*}}} \tag{7.53}$$

where $(\partial g/\partial X'_i)^*$ is the ith partial derivative evaluated at the design point with coordinates $(x_1'^*, x_2'^*, \ldots, x_n'^*)$. The asterisk after the derivative indicates that it is evaluated at $(x_1'^*, x_2'^*, \ldots, x_n'^*)$. The design point in the reduced coordinates is given by:

$$x_i'^* = -\alpha_i \beta_{HL} \quad (i = 1, 2, \ldots, n) \tag{7.54}$$

where

$$\alpha_i = \frac{\left(\frac{\partial g}{\partial X_i'}\right)^*}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i'}\right)^{2^*}}} \tag{7.55}$$

are the direction cosines along the coordinate axes X_i'. In the space of the original coordinates and using Equation 7.45, we find the design point to be

$$x_i^* = \mu_{X_i} - \alpha_i \sigma_{X_i} \beta_{HL}. \tag{7.56}$$

An algorithm was formulated by Rackwitz (1976) to compute β_{HL} and $x_i'^*$ as follows:

- Step 1. Define the appropriate limit state equation.
- Step 2. Assume initial values of the design point x_i^*, $i = 1, 2, \ldots, n$. Typically, the initial design point may be assumed to be at the mean values of the random variables. Obtain the reduced variates $x_i'^* = (x_i^* - \mu_{X_i}) / \sigma_{X_i}$.
- Step 3. Evaluate $(\partial g/\partial X_i')^*$ and α_i at $x_i'^*$.

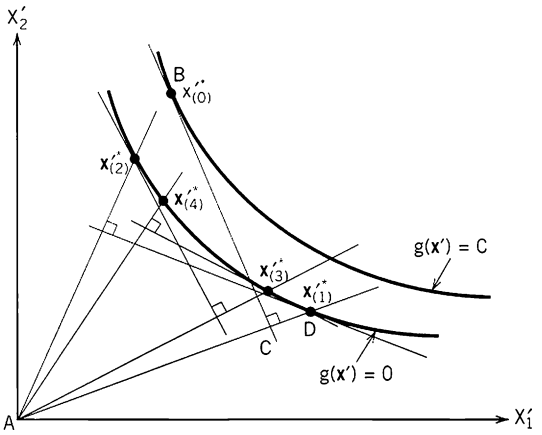
- Step 4. Obtain the new design point $x_i'^*$, in terms of β_{HL}, as in Equation 7.54.
- Step 5. Substitute the new $x_i'^*$ in the limit state equation $g(\mathbf{x}'^*) = 0$ and solve for β_{HL}.
- Step 6. Using the β_{HL} value obtained in Step 5, reevaluate $x_i'^* = -\alpha_i \beta_{HL}$.
- Step 7. Repeat Steps 3 through 6 until β_{HL} converges.

This algorithm is shown geometrically in Figure 7.6. The algorithm constructs a linear approximation to the limit state at every search point and finds the distance from the origin to the limit state. In Figure 7.6, Point B represents the initial design point, usually assumed to be at the mean values of the random variables, as noted in Step 2. Note that B is not on the limit state equation $g(\mathbf{X}') = 0$. The tangent to the limit state at B is represented by the line BC. Then AD will give an estimate of β_{HL} in the first iteration, as noted in Step 5. As the iteration continues, β_{HL} value converges.

Ditlevsen (1979a) showed that for a nonlinear limit state surface, β_{HL} lacks comparability; the ordering of β_{HL} values may not be consistent with the ordering of actual reliabilities. An example of this is shown in Figure 7.5 with two limit state surfaces: one flat and the other curved. The shaded region to the right of each limit state represents the corresponding failure region. Clearly, the structure with the flat limit state surface has a different reliability than the one with the curved limit state surface; however, the β_{HL} values are identical for both surfaces and suggest equal reliability. To overcome this inconsistency, Ditlevsen (1979a) introduced the generalized reliability index, β_g, defined as

$$\beta_g = \Phi^{-1}\left[\int_{g(\mathbf{x}')>0} \cdots \int \phi(x_1')\phi(x_2') \cdots \phi(x_n')dx_1'dx_2' \cdots dx_n'\right] \tag{7.57}$$

where Φ and ϕ are the cumulative distribution function and the probability density function of a standard normal variable, respectively. Because the reliability index in this definition includes the entire safe region, it provides a consistent ordering of second-moment reliability. The integral in the equation looks similar to that in Equation 7.35 and is difficult to compute directly. Hence, Ditlevsen (1979a) proposed approximating the nonlinear limit state by a polyhedral surface consisting of tangent hyperplanes at selected points on the surface.



Note: A number in parentheses indicates iteration number.

Figure 7.6 Algorithm for Finding β_{HL}

EXAMPLE 7.4

Denoting R and S as the random variables representing the resistance and the applied load on a structure, assume that the limit state equation is represented by Equation 7.47 in the original coordinate system and by Equation 7.50 in the reduced coordinate system. These are shown in Figures 7.4a and 7.4b, respectively. Using Equation 7.55, we can evaluate the direction cosines α_R and α_S as

$$\alpha_R = \frac{(\partial g / \partial R')}{\sqrt{(\partial g / \partial R')^2 + (\partial g / \partial S')^2}} = \frac{\sigma_R}{\sqrt{\sigma_R^2 + \sigma_S^2}}$$

and

$$\alpha_S = -\frac{\sigma_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}.$$

Notice that in Figure 7.4b, $\alpha_R = \cos \theta_R$ and $\alpha_S = \cos \theta_S$. The angles θ_R and θ_S are defined as counterclockwise angles of rotation from the positive directions of R' and S' axes to the positive direction of the $\boldsymbol{\alpha}$ vector (i.e., vector of direction cosines). Using Equation 7.54, we can show the coordinates of the checking point in the reduced coordinate to be

$$r'^* = -\alpha_R \beta_{HL} = -\frac{\sigma_R}{\sqrt{\sigma_R^2 + \sigma_S^2}} \beta_{HL}$$

$$s'^* = -\alpha_S \beta_{HL} = \frac{\sigma_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \beta_{HL}.$$

Substituting these new checking points in Equation 7.50, we can calculate the reliability index β_{HL} as

$$\beta_{HL} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}.$$

Since it is a linear limit state equation, iteration is not required. This is the same result as obtained by Equation 7.36, indicating that the algorithm works correctly. Using Equation 7.56, we find the new checking point in the original coordinates to be

$$r^* = \mu_R - \left(\frac{\sigma_R}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right)\sigma_R \left(\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right) = \frac{\mu_R \sigma_S^2 + \mu_S \sigma_R^2}{\sigma_R^2 + \sigma_S^2}$$

$$s^* = \mu_S - \left(-\frac{\sigma_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right)\sigma_S \left(\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right) = \frac{\mu_R \sigma_S^2 + \mu_S \sigma_R^2}{\sigma_R^2 + \sigma_S^2}.$$

In this case, $r^* = s^*$, indicating that the checking point is on the limit state line, which is at a 45° angle to both coordinate axes.

Several important observations can be made by comparing the safety indices calculated by the MVFOSM (Equation 7.36) and the AFOSM proposed by Hasofer and Lind (Equation 7.53). As long as the limit state equation of resistance and load is linear and all the variables are normal, the safety indices calculated by the two methods will be the same. However, strictly speaking, the MVFOSM does not use any information on the distribution of the resistance and load, whereas the AFOSM proposed by Hasofer and Lind is applicable when they are normal. The most important difference is that in the

MVFOSM method, the design point is at the mean values of R and S, indicating they are not on the limit state line. The AFOSM (Hasofer–Lind) method indicates that the design point is on the limit state line. This can be elaborated further with the help of an example.

EXAMPLE 7.5

Suppose a cable of resistance R needs to carry a weight S. Assume both R and S are normal random variables with means of 120 kip and 50 kip, respectively, and corresponding standard deviations of 18 kip and 12 kip, respectively. The limit state equation can be represented by Equation 7.47. Then, the safety index according to the MVFOSM and the Hasofer–Lind methods will be the same, that is,

$$\beta = \beta_{HL} = \frac{120 - 50}{\sqrt{18^2 + 12^2}} = 3.236.$$

The design point according to the MVFOSM is (120, 50) as shown in Figure 7.4a. The coordinates of the design point according to the Hasofer–Lind method can be estimated as

$$r^* = s^* = \frac{\mu_R \sigma_S^2 + \mu_S \sigma_R^2}{\sigma_R^2 + \sigma_S^2} = \frac{120 \times 12^2 + 50 \times 18^2}{18^2 + 12^2} = 71.54.$$

The coordinates for the checking point are (71.54, 71.54), indicating that it is on the limit state equation as shown in Figure 7.4b.

7.6.3 AFOSM Methods for Nonnormal Variables

The Hasofer–Lind reliability index can be exactly related to the failure probability using Equation 7.37 if all the variables are statistically independent and normally distributed and the limit state surface is linear. For any other situation, it will not give correct information on the probability of failure. Rackwitz and Fiessler (1978), Chen and Lind (1983), and others corrected this shortcoming and included information on the distributions of the random variables in the algorithm for both the linear and nonlinear limit state equations. In the context of AFOSM, the probability of failure has been estimated using two types of approximations to the limit state at the design point: first order (leading to the name FORM) and second order (leading to the name SORM). The MVFOSM discussed in Section 7.6.1 is an earlier version of FORM. At the present time, AFOSM is known as FORM. The Hasofer–Lind method discussed in Section 7.6.2 is an earlier version of AFOSM, applicable when all the variables are normal. Other FORM methods are discussed next, and SORM is discussed in Chapter 8.

7.6.3.1 Equivalent Normal Variables

The deficiency in the Hasofer–Lind method, that it is applicable only for normal variables, needs to be addressed at this stage. If not all the variables are normally distributed, as is common in engineering problems, it is necessary to transform the nonnormal variables into equivalent normal variables. The *Rosenblatt transformation* (Rosenblatt, 1952) can be used to obtain a set of statistically independent standard normal variables, if the joint CDF of all the random variables is available. Conceptually, statistically independent nonnormal variables can be transformed to equivalent normal variables in

several ways. Procedures to transform correlated nonnormal variables are discussed in Chapter 8. Because a normal random variable can be described uniquely by two parameters (mean and standard deviation), any two appropriate conditions can be used for this purpose. Paloheimo (1973) suggested approximating a nonnormal distribution by a normal distribution having the same mean value and the same P percentile (the value of the variate at which the cumulative probability is $P\%$). He set P equal either to the target failure probability p_f if the variable was a loading variable or to $(1.0 - p_f)$ if the variable was a resistance variable. The *Rackwitz–Fiessler method* (two-parameter equivalent normal), the *Chen–Lind method*, and the *Wu–Wirsching method* (three-parameter equivalent normal) can also be used for this purpose and are discussed next.

7.6.3.2 Two-Parameter Equivalent Normal Transformation

Rackwitz and Fiessler (1976) estimated the parameters of the equivalent normal distribution, $\mu_{X_i}^N$ and $\sigma_{X_i}^N$, by imposing two conditions. The cumulative distribution functions and the probability density functions of the actual variables and the equivalent normal variables should be equal at the checking point ($x_1^*, x_2^*, \ldots, x_n^*$) on the failure surface. Considering each statistically independent nonnormal variable individually and equating its CDF with an equivalent normal variable at the checking point results in

$$\Phi\left(\frac{x_i^* - \mu_{X_i}^N}{\sigma_{X_i}^N}\right) = F_{X_i}(x_i^*) \tag{7.58}$$

in which $\Phi(\,)$ is the CDF of the standard normal variate, $\mu_{X_i}^N$ and $\sigma_{X_i}^N$ are the mean and standard deviation of the equivalent normal variable at the checking point, and $f_{X_i}(x_i^*)$ is the CDF of the original nonnormal variables. Equation 7.58 yields

$$\mu_{X_i}^N = x_i^* - \Phi^{-1}[F_{X_i}(x_i^*)]\sigma_{X_i}^N. \tag{7.59}$$

Equating the PDFs of the original variable and the equivalent normal variable at the checking point results in

$$\frac{1}{\sigma_{X_i}^N}\phi\left(\frac{x_i^* - \mu_{X_i}^N}{\sigma_{X_i}^N}\right) = f_{X_i}(x_i^*) \tag{7.60}$$

in which $\phi(\,)$ and $f_{X_i}(x_i^*)$ are the PDFs of the equivalent standard normal and the original nonnormal random variable. Equation 7.60 yields

$$\sigma_{X_i}^N = \frac{\phi\left\{\Phi^{-1}\left[F_{X_i}(x_i^*)\right]\right\}}{f_{X_i}(x_i^*)}. \tag{7.61}$$

Having determined $\mu_{X_i}^N$ and $\sigma_{X_i}^N$ and proceeding similarly to the case in which all random variables are normal, we can obtain β_{HL} using the 7 steps described earlier. Then Equation 7.37 can be used to calculate the failure probability. This approach became well known as the Rackwitz–Fiessler method and has been used extensively in the literature.

This approximation of nonnormal distributions can become more and more inaccurate if the original distribution becomes increasingly skewed. For highly skewed distributions, such as the Fréchet (Type II distribution of maxima, see Section 4.5.5),

the conditions represented in Equations 7.59 and 7.61 need to be modified. In this case, the mean value and the probability of exceedence of the equivalent normal variable are made equal to the median value and the probability of exceedence of the original random variable, respectively, at the checking point (Rackwitz and Fiessler, 1978). $\mu_{X_i}^N$ and $\sigma_{X_i}^N$ can be estimated as

$$\mu_{X_i}^N = F_{X_i}^{-1}(0.5) = \text{median of } X_i \tag{7.62}$$

and

$$\sigma_{X_i}^N = \frac{x_i^* - \mu_{X_i}^N}{\Phi^{-1}[F_{X_i}(x_i^*)]} \tag{7.63}$$

in which $F_{X_i}^{-1}(\,)$ is the inverse of the nonnormal CDF of X_i.

For highly skewed random variables, usually load-related variables, and relatively large values of x_i*, the cumulative distribution function at x_i* will be close to one, and the value of the density function at x_i* will be very small. Rackwitz and Fiessler (1978) observed, as we did, that if Equations 7.59 and 7.61 are used to calculate $\mu_{X_i}^N$ and $\sigma_{X_i}^N$, $\mu_{X_i}^N$ will be forced to be small. The larger x_i* is, the smaller $\mu_{X_i}^N$ will tend to be. But this may destroy the validity of the distribution of X_i; for example, for the Fréchet distribution; it is only valid for the positive values of the random variable. As shown by Ayyub and Haldar (1984), this problem might occur in many designs. A lower limit on $\mu_{X_i}^N$ of zero is suggested and has been proven to give accurate estimates of β and p_f using the optimization algorithm of FORM and SORM. If this lower value is imposed on $\mu_{X_i}^N$, then if $\mu_{X_i}^N < 0$,

$$\sigma_{X_i}^N = \frac{x_i^*}{\Phi^{-1}[F_{X_i}(x_i^*)]} \tag{7.64}$$

and

$$\mu_{X_i}^N = 0; \tag{7.65}$$

otherwise use Equations 7.59 and 7.61.

The 7 steps described in Section 7.6.2 to calculate β_{HL} are still applicable for the Rackwitz–Fiessler method if all the random variables in the limit state equation are normal. If some or all of them are not normal random variables, then another step is necessary. In this step the equivalent normal mean and standard deviation of all the nonnormal random variables at the design point need to be estimated.

Two optimization algorithms are commonly used to obtain the design point and the corresponding reliability or safety index. The first method (Rackwitz, 1976) requires solution of the limit state equation during the iterations and will be referred to as FORM Method 1 in the following discussion. The second method (Rackwitz and Fiessler, 1978) does not require solution of the limit state equation. Instead, it uses a Newton-type recursive formula to find the design point. This method will be referred to as FORM Method 2 in the subsequent discussion.

7.6.3.3 FORM Method 1

The steps in this method to estimate the reliability or safety index are explained as follows including the computation of parameters for equivalent normal variables. Some improvements in the algorithm suggested by Ayyub and Haldar (1984) are included in these steps. The original coordinate system is used in describing these steps.

- Step 1. Define the appropriate limit state equation.
- Step 2. Assume an initial value of the safety index β. Any value of β can be assumed; if it is chosen intelligently, the algorithm will converge in a very few steps. An initial β value of 3.0 is reasonable.
- Step 3. Assume the initial values of the design point x_i^*, $i = 1, 2, \ldots, n$. In the absence of any other information, the initial design point can be assumed to be at the mean values of the random variables.
- Step 4. Compute the mean and standard deviation at the design point of the equivalent normal distribution for those variables that are nonnormal.
- Step 5. Compute partial derivatives $(\partial g/\partial X_i)^*$ evaluated at the design point x_i^*.
- Step 6. Compute the direction cosines α_i at the design point as

$$\alpha_{X_i} = \frac{\left(\frac{\partial g}{\partial X_i}\right)^* \sigma_{X_i}^N}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\sigma_{X_i}^N\right)^{2^*}}} \tag{7.66}$$

Note that Equations 7.55 and 7.66 are identical. In Equation 7.55, the direction cosines are evaluated in the reduced coordinates where the standard deviations of the reduced variables are unity. In Equation 7.66, if the random variables are normal, then their standard deviations can be used directly; otherwise, for nonnormal random variables, the equivalent standard deviations at the checking point need to be used.

- Step 7. Compute the new values for checking point x_i^* as

$$x_i^* = \mu_{X_i}^N - \alpha_i \beta \sigma_{X_i}^N \tag{7.67}$$

  If necessary, repeat Steps 4 through 7 until the estimates of α_i converge with a predetermined tolerance. A tolerance level of 0.005 is common. Once the direction cosines converge, the new checking point can be estimated, keeping β as the unknown parameter. This additional computation may improve the robustness of the algorithm. Note that the assumption of an initial value for β in Step 2 is necessary only for the sake of this additional computation. Otherwise, Step 2 can be omitted.
- Step 8. Compute an updated value for β using the condition that the limit state equation must be satisfied at the new checking point.
- Step 9. Repeat Steps 3 through 8 until β converges to a predetermined tolerance level. A tolerance level of 0.001 can be used, particularly if the algorithm is developed in a computer environment.

The algorithm converges very rapidly, most of the time within 5 to 10 cycles, depending upon the nonlinearity in the limit state equation. A small computer program can be written to carry out the necessary calculations.

EXAMPLE 7.6

To help implement the preceding algorithm, an example of a detailed step-by-step solution is given next.

Example 7.3, summarized in Table 7.1, is considered again. To estimate the safety index using FORM, the distributions of F_y and Z need to be considered. For illustration purposes, assume that F_y is a lognormal variable with a mean of 38 ksi and standard deviation of 3.8 ksi, and Z is a normal random variable with a mean of 54 in.^{3} and standard deviation of 2.7 in.^{3} The strength limit state is considered in this example. The 9 steps necessary to estimate the safety index using FORM are summarized in Table 7.2. For ease of comprehension, these steps are explained in detail as follows.

- Step 1. Using the strength formulation, we can express the limit state equation for the problem as $g(\,) = F_y Z - 1{,}140 = 0$.
- Step 2. Assume $\beta = 3.0$.
- Step 3. The initial design point is assumed to be 38 and 54, the mean values of F_y and Z, respectively.
- Step 4. Since Z is a normal random variable, no additional transformations are needed. However, since F_y is a lognormal variable, its equivalent normal mean and standard deviation at the design point can be estimated in two ways, as discussed next.

Alternative 1 Using Equations 7.59 and 7.61

In this case, $\delta_{F_y} = 0.1$. Thus,

$$\zeta_{F_y} = \sqrt{\ln(1+\delta_{F_y}^2)} = \sqrt{\ln(1+0.1^2)} = 0.0997513$$

Table 7.2 Steps in FORM Method 1

Step 1				$g(\,) = F_y Z - 1{,}140$			
Step 2	β	3.0			5.002		5.150
Step 3	f_y^*	38.	27.64	29.02	23.96	24.50	24.21
	z^*	54.	50.37	50.17	47.59	47.32	47.10
Step 4	$\mu_{F_y}^N$	37.81	36.30	36.70	34.89	35.13	35.00
	$\sigma_{F_y}^N$	3.79	2.76	2.89	2.39	2.44	2.42
	μ_Z^N	54.0	54.0	54.0	54.0	54.0	54.0
	σ_Z^N	2.7	2.7	2.7	2.7	2.7	2.7
Step 5	$\left(\frac{\partial g}{\partial F_y}\right)^*$	54.0	50.37	50.17	47.59	47.32	47.10
	$\left(\frac{\partial g}{\partial Z}\right)^*$	38.0	27.64	29.02	23.96	24.50	24.21
Step 6	α_{F_y}	0.894	0.881	0.880	0.869	0.868	0.867
	α_Z	0.448	0.473	0.475	0.494	0.496	0.498
Step 7		Go to Step 3. Compute the new checking point using information from Step 6.					
Step 8	β			5.002		5.150	5.151
Step 9		Repeat Steps 3 through 8 until β converges.					

The final checking point is (24.22, 47.07).

and

$$\lambda_{F_y} = \ln \mu_{F_y} - \frac{1}{2}\zeta_{F_y}^2 = \ln 38 - \frac{1}{2}(0.0997513)^2 = 3.632611$$

$$f_{F_y}(f_y^*) = f_{F_y}(38) = \frac{1}{\sqrt{2\pi}\zeta_{F_y} f_y^*} \exp\left[-\frac{1}{2}\left(\frac{\ln f_y^* - \lambda_{F_y}}{\zeta_{F_y}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}(0.0997513)(38)} \exp\left[-\frac{1}{2}\left(\frac{\ln 38 - 3.632611}{0.0997513}\right)^2\right] = 0.1051157$$

$$F_{F_y}(f_y^*) = P(0 < F_y \le 38) = \Phi\left(\frac{\ln 38 - 3.632611}{0.0997513}\right) = \Phi(0.0498756).$$

Thus,

$$\Phi^{-1}\left[F_{F_y}(f_y^*)\right] = 0.0498756$$

$$\phi\left\{\Phi^{-1}\left[F_{F_y}(f_y^*)\right]\right\} = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(0.0498756)^2\right] = 0.3984464.$$

Using Equations 7.61, we can show that

$$\sigma_{F_y}^N = \frac{0.3984464}{0.1051157} = 3.7905487.$$

Using Equation 7.59, we can show that

$$\mu_{F_y}^N = 38 - (0.0498756)(3.7905487) = 37.810944.$$

Alternative 2 (Simplified Approach)

For a lognormal random variable X with parameters λ_X and ζ_X, the equivalent normal mean and standard deviation at the design point x^* can be shown to be

$$\sigma_X^N = \zeta_X x^* \tag{7.68}$$

and

$$\mu_X^N = x^*(1 - \ln x^* + \lambda_X). \tag{7.69}$$

Thus,

$$\sigma_{F_y}^N = \zeta_{F_y} f_y^* = (0.0997513)(38) = 3.7905494$$

and

$$\mu_{F_y}^N = 38(1 - \ln 38 + 3.632611) = 37.810944.$$

These are the same values estimated using Alternative 1. The information is summarized in Table 7.2.

- Step 5. For the example under consideration, the partial derivatives $(\partial g/\partial X_i)^*$ evaluated at the design point can be shown to be

$$\left(\frac{\partial g}{\partial F_y}\right)^* = z^* = 54 \quad \text{and} \quad \left(\frac{\partial g}{\partial Z}\right)^* = f_y^* = 38.$$

- Step 6. Equation 7.66 can be used to calculate the direction cosines for F_y and Z:

$$\alpha_{F_y} = \frac{54 \times 3.7905487}{\sqrt{(54 \times 3.7905487)^2 + (38 \times 2.7)^2}} = 0.8939809$$

$$\alpha_Z = \frac{38 \times 2.7}{\sqrt{(54 \times 3.7905487)^2 + (38 \times 2.7)^2}} = 0.4481049.$$

- Step 7. Equation 7.67 is used to find the coordinates of the new design point:

$$f_y^* = 37.810944 - 0.8939809 \times 3.0 \times 3.7905487 = 27.644908$$

and

$$z^* = 54 - 0.4481049 \times 3.0 \times 2.7 = 50.37035.$$

The second iteration will start with the coordinates of the new design point just calculated in Step 7, as shown in Table 7.2. Steps 3 through 7 are repeated until the direction cosines converge at a tolerance level of 0.005. The detailed calculations are not shown here; however, they are similar to the calculations just discussed. At the third iteration, α_{F_y} and α_Z converge to 0.8800674 and 0.4748486, respectively.

- Step 8. The coordinates of the new design point, keeping β as the unknown parameter, are

$$f_y^* = 36.698103 - (0.8800674)(2.8942793)\beta = 36.698103 - 2.54716\beta$$

and

$$z^* = 54 - (0.4748486)(2.7)\beta = 54 - 1.28209\beta.$$

A new β value can be estimated by satisfying the limit state equation as

$$(36.698103 - 2.54716\beta)(54 - 1.28209\beta) - 1,140 = 0.$$

When this equation is solved, β is found to be 5.002. This updated β is considerably different than the initial assumed value of 3.0. With the updated β value, the coordinates on the new design point become

$$f_y^* = 36.70 - 0.880 \times 5.002 \times 2.89 = 23.96$$

and

$$z^* = 54 - 0.475 \times 5.002 \times 2.7 = 47.59.$$

Thus, the fourth iteration will start with the updated information on the coordinates of the design point, as shown in Table 7.2. Again, the direction cosines converge after the fifth iteration, and the updated β becomes 5.150.

- Step 9. Steps 3 through 8 are repeated until β converges to a tolerance level of 0.005. As shown in Table 7.2, at the sixth iteration, β converges to 5.151 with a tolerance level of 0.005, and the corresponding checking point is (24.22, 47.07).

7.6.3.4 FORM Method 2

Notice that in Step 8 of FORM Method 1, the limit state equation needs to be solved to find the new design point. This may be difficult in the case of complicated nonlin-

ear g-functions. Also, in many practical problems, the g-function may not even be available in closed form. In that case, it is impossible to perform Step 8, thus limiting the usefulness of FORM Method 1. Therefore, an alternative Newton–Raphson type recursive algorithm, referred to as FORM Method 2 in this section, is presented here to find the design point. This algorithm, suggested by Rackwitz and Fiessler (1978), is similar to FORM Method 1 in that it linearizes the performance function at each iteration point; however, instead of solving the limit state equation explicitly for β, it uses the derivatives to find the next iteration point.

The algorithm can best be explained with the help of Figures 7.7 and 7.8. Consider first the linear performance function shown in Figure 7.7. Since the limit state is not available in closed form, the starting point $\mathbf{x}_0'$ (usually the vector of mean values of the random variables) may not be on the limit state $g(X_1', X_2') = 0$, but on a parallel line $g(X_1', X_2') = k$. Hence the optimization algorithm has to start from point $\mathbf{x}_0'$ which may not be on the limit state, and converge to the minimum distance point $\mathbf{x}'^*$ on the limit state. The linear performance function $g(\mathbf{x}')$ may be expressed as

$$\begin{aligned} g(\mathbf{x}') &= b + \mathbf{a}^t \mathbf{x}' \\ &= b + a_1 x_1' + a_2 x_2'. \end{aligned} \tag{7.70}$$

Here $\mathbf{a}^t = (a_1, a_2)$ is the transpose of the gradient vector (i.e., vector of first derivatives) of the performance function. The magnitudes of the vectors $\mathbf{x}_0'^*$ and $\mathbf{x}'^*$ denote the distance from the origin to starting point and to the limit state $g(\mathbf{x}') = 0$, respectively. Using geometry, $\mathbf{x}'^*$ can be expressed in terms of $\mathbf{x}_0'^*$ as

$$\mathbf{x}'^* = \frac{1}{|\mathbf{a}|^2}[\mathbf{a}^t \mathbf{x}_0'^* - g(\mathbf{x}_0'^*)]\{\mathbf{a}\} \tag{7.71a}$$

Rewriting Equation 7.71a in terms of the components of all the vectors results in

$$\begin{Bmatrix} x_1'^* \\ x_2'^* \end{Bmatrix} = \frac{1}{a_1^2 + a_2^2}[a_1 x_{01}'^* + a_2 x_{02}'^* - g(x_{01}'^*, x_{02}'^*)]\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \tag{7.71b}$$

Since the performance function is linear in this case, its gradient is constant; hence the distance to the limit state from the origin is obtained in one step.

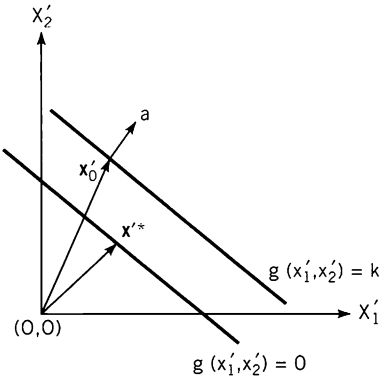


Figure 7.7 FORM Method 2 for a Linear Performance Function

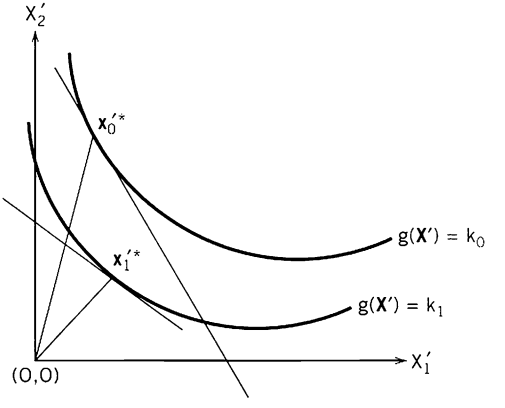


Figure 7.8 FORM Method 2 for a Nonlinear Performance Function

Equation 7.71a can be generalized for a nonlinear performance function as shown in Figure 7.8 as

$$\mathbf{x}'^{*}_{k+1} = \frac{1}{|\nabla g(\mathbf{x}'^{*}_{k})|^2}\left[\nabla g(\mathbf{x}'^{*}_{k})^t \mathbf{x}'^{*}_{k} - g(\mathbf{x}'^{*}_{k})\right]\nabla g(\mathbf{x}'^{*}_{k}) \tag{7.72}$$

where $\nabla g(\mathbf{x}'^{*}_{k})$ is the gradient vector of the performance function at $\mathbf{x}'^{*}_{k}$, the kth iteration point. Note that k refers to the iteration number. Therefore $\mathbf{x}'^{*}_{k}$ is a vector with components $\{x'^{*}_{1k}, x'^{*}_{2k}, \ldots, x'^{*}_{nk}\}^t$, where n is the number of random variables. The meaning of $\mathbf{x}'^{*}_{k+1}$ is similar.

Since the performance function is nonlinear, the gradient is not constant but varies from point to point. Therefore, instead of a one-step solution in the case of the linear performance function, the point of minimum distance has to be searched through the recursive formula given in Equation 7.72. This formula can be geometrically interpreted using Figure 7.8. At each iteration point, the performance function is approximated by the tangent at the point, that is, the performance function is linearized with $g(\mathbf{x}'^{*}_{k})$ and $\nabla g(\mathbf{x}'^{*}_{k})$ corresponding to $g(\mathbf{x}'^{*}_{0})$ and $\mathbf{a}$, respectively, in Equation 7.71a. The next iteration point $\mathbf{x}'^{*}_{k+1}$ is computed the same way as in the case of the linear performance function. If the performance function were linear, $\mathbf{x}'^{*}_{k+1}$ would be identical to $\mathbf{x}'^{*}_{k}$, for $k > 0$. However, since the performance function is nonlinear, its value and gradient at $\mathbf{x}'^{*}_{k-1}$ are different from those at $\mathbf{x}'^{*}_{k}$. Therefore it is again linearized at $\mathbf{x}'^{*}_{k+1}$ and another iteration point $\mathbf{x}'^{*}_{k+2}$ is computed. The algorithm is repeated until convergence, satisfying the following two criteria:

$$\text{If } |\mathbf{x}'^{*}_{k} - \mathbf{x}'^{*}_{k-1}| \le \delta, \text{ stop.}$$

$$\text{If } |g(\mathbf{x}'^{*}_{k})| \le \varepsilon, \text{ stop.}$$

Both δ and ε are small quantities, say 0.001.

From this discussion, it is obvious that the recursive formula in Equation 7.72 results from the linearization of the performance function. Consider a first-order Taylor series approximation of the performance function as

$$g(\mathbf{x}'^{*}_{k+1}) = g(\mathbf{x}'^{*}_{k}) + \nabla g(\mathbf{x}'^{*}_{k})(\mathbf{x}'^{*}_{k+1} - \mathbf{x}'^{*}_{k}). \tag{7.73}$$

Thus, the limit state $g(\mathbf{x}'^{*}_{k+1}) = 0$ becomes

$$g(\mathbf{x}'^{*}_{k}) + \nabla g(\mathbf{x}'^{*}_{k})(\mathbf{x}'^{*}_{k+1} - \mathbf{x}'^{*}_{k}) = 0. \tag{7.74}$$

Rearrangement of the terms in this equation gives Equation 7.72 as the solution for the minimum distance point on the linearized limit state. (The distance is measured from the origin.)

Compared to other nonlinear optimization algorithms available in the literature, the algorithm just described requires the least computation at each step. The next iteration point is computed using a single recursive formula that requires information only about the value and the gradient of the performance function. The storage requirement is therefore minimal. The algorithm is also found to converge fast in many cases. For these reasons, this algorithm has been widely used in the literature.

Convergence Problems

This algorithm may fail to converge in some situations. It may converge very slowly, or oscillate about the solution without convergence, or diverge away from the solution. Two such examples are shown in Figures 7.9 and 7.10. For the case of a single variable x', the formula in Equation 7.72 reduces to the Newton–Raphson method to find the root of $g(x') = 0$ (see Equation 7.74). It is well known that the Newton–Raphson formula may fail to find the roots of a function in certain circumstances. Figure 7.9 illustrates one such situation where the Newton–Raphson method diverges further and further away from the solution. In the example with two variables (Figure 7.10), the performance function is

$$g(\mathbf{x}') = x'_1 x'_2 - d. \tag{7.75}$$

If the starting point (a, b) falls on one of the two ellipses $(x'_1)^2 + (x'_2)^2 + x'_1 x'_2 + d = 0$ and $(x'_1)^2 + (x'_2)^2 - x'_1 x'_2 - d = 0$, then the algorithm generates points that oscillate between (a, b) and (b, a), as shown by Liu and Der Kiureghian (1986).

Thus, it is possible that the Rackwitz–Fiessler algorithm may not converge to the MPP (minimum distance point, or most probable point of failure) in some cases. Other optimization algorithms such as sequential quadratic programming or the BFGS (Broyden–Fletcher–Goldfarb–Shanno) method (Vanderplaats, 1984) may be used in that case.

Similar to FORM Method 1, FORM Method 2 can be described as follows. Both the original and equivalent standard normal or reduced coordinate systems are used in this method.

- Step 1. Define the appropriate performance function.
- Step 2. Assume initial values of the design point $\mathbf{x}_i^*$, $i = 1, 2, \ldots, n$, and compute the corresponding value of the performance function $g(\,)$. In the absence of any other information, the initial design point can be the mean values of the random variables.
- Step 3. Compute the mean and standard deviation at the design point of the equivalent normal distribution for those variables that are nonnormal. The coordinates of the design point in the equivalent standard normal space are

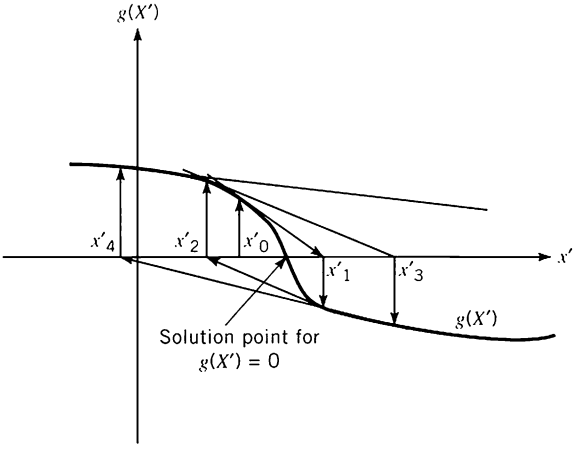


Figure 7.9 Example of Failure of the Newton–Raphson Method

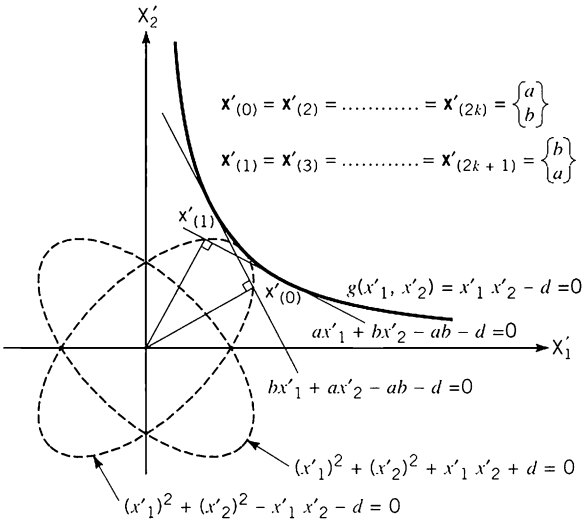


Figure 7.10 Nonconvergence of the Rackwitz–Fiessler Algorithm—Bivariate Case

$$x_i'^* = \frac{x_i' - \mu_{X_i}^N}{\sigma_{X_i}^N}. \tag{7.76}$$

- Step 4. Compute the partial derivative $\partial g/\partial X_i$ evaluated at the design point $\mathbf{x}_i^*$.
- Step 5. Compute the partial derivatives $\partial g/\partial X'_i$ in the equivalent standard normal space by the chain rule of differentiation as

$$\frac{\partial g}{\partial X_i'} = \frac{\partial g}{\partial X_i}\frac{\partial X_i}{\partial X_i'} = \frac{\partial g}{\partial X_i}\sigma_{X_i}^N. \tag{7.77}$$

The partial derivatives $\partial g/\partial X'_i$ are the components of the gradient vector of the performance function in the equivalent standard normal space. The components of the corresponding unit vector are the direction cosines of the performance function, computed as

$$\alpha_i = \frac{\left(\frac{\partial g}{\partial X_i'}\right)^*}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i'}\right)^{2^*}}} = \frac{\left(\frac{\partial g}{\partial X_i}\right)^*\sigma_{X_i}^N}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\sigma_{X_i}^N\right)^{2^*}}}. \tag{7.78}$$

Note that this is exactly the same formula as in Equation 7.66. Although the direction cosines are not directly used in the current algorithm, they are used later in the implementation of SORM, the second-order reliability method, discussed in Chapter 8.

- Step 6. Compute the new values for the design point in the equivalent standard normal space ($\mathbf{x}'_i{}^*$) using the recursive formula of Equation 7.72.
- Step 7. Compute the distance to this new design point from the origin as

$$\beta = \sqrt{\sum_{i=1}^{n}(x_i'^*)^2}. \tag{7.79}$$

Check the convergence criterion for β (i.e., the change in the value of β between two consecutive iterations is less than a predetermined tolerance level, say 0.001).

- Step 8. Compute the new values for the design point in the original space (x_i^*) as

$$x_i^* = \mu_{X_i}^N + \sigma_{X_i}^N x_i'^* \tag{7.80}$$

Compute the value of the performance function $g(\ )$ for this new design point, and check the convergence criterion for $g(\ )$; that is, check that the value of $g(\ )$ is very close to zero, say within 0.001. If both convergence criteria are satisfied, stop. Otherwise, repeat Steps 3 through 8 until convergence.

EXAMPLE 7.7

Consider again the performance function $g(\ ) = F_y Z - 1{,}140$ used to demonstrate FORM Method 1 in Example 7.6. Note that FORM Method 2 is particularly useful when the performance function is implicit, that is, when it cannot be written as a closed-form expression in terms of the random variables. However, this simple closed-form performance function is chosen for the sake of illustration and comparison with FORM Method 1. Issues related to implicit functions are discussed in detail by the authors in another book titled *Reliability Assessment Using Stochastic Finite Element Analysis*, published by John Wiley & Sons in 2000.

F_y is assumed to have a lognormal distribution with a mean value of 38.0 ksi and a standard deviation of 3.8 ksi. Z is assumed to have a normal distribution with a mean value of 54.0 in.^{3} and a standard deviation of 2.7 in.^{3}. The 8 steps of FORM Method 2 are summarized in Table 7.3. For ease of comprehension, the first iteration is discussed here.

- Step 1. The performance function is $g(\ ) = F_y Z - 1{,}140$.
- Step 2. The initial values of the design point are chosen to be the same as the mean values of the two random variables, that is, $f_y^* = 38$ and $z^* = 54$. For this initial design point, the value of $g(\ )$ is computed as

$$g(\ ) = (38)(54) - 1{,}140 = 912.$$

- Step 3. The equivalent normal mean and standard deviation for the lognormal variable F_y are computed in the same way as in FORM Method 1 as

$\mu_{F_y}^N = 37.81$ and $\sigma_{F_y}^N = 3.79$. Since Z is a normal random variable, its equivalent mean and standard deviation are the same as the original mean and standard deviation. Using Equation 7.76, the coordinates of the design point in the equivalent standard normal space are

$$f_y'^* = \frac{38 - 37.81}{3.79} = 0.05, \quad z'^* = \frac{27 - 27}{2.7} = 0.$$

- Step 4. The partial derivatives evaluated at the design point are

$$\left(\frac{\partial g}{\partial F_y}\right)^* = z^* = 54 \text{ and } \left(\frac{\partial g}{\partial Z}\right)^* = f_y^* = 38.$$

Table 7.3 Steps in FORM Method 2

Step 1		$g(\,) = F_y Z - 1{,}140$			
Step 2		Initial values: $f_y^* = 38$, $z^* = 54$, $g(\,) = 912.0$			
Step 3	$\mu_{F_y}^N$	37.81	35.116	34.960	35.003
	$\sigma_{F_y}^N$	3.79	2.44	2.405	2.415
	μ_Z^N	54.00	54.00	54.00	54.00
	σ_Z^N	2.70	2.70	2.70	2.70
	$f_y'^*$	0.05	–4.365	–4.510	–4.471
	z'^*	0.00	–1.765	–2.479	–2.558
Step 4	$\left(\frac{\partial g}{\partial F_y}\right)^*$	54.00	49.235	47.307	47.093
	$\left(\frac{\partial g}{\partial Z}\right)^*$	38.00	24.464	24.112	24.207
Step 5	$\left(\frac{\partial g}{\partial F_y'}\right)^*$	204.69	120.15	113.78	113.71
	$\left(\frac{\partial g}{\partial Z'}\right)^*$	102.60	66.05	65.10	65.36
Step 6	New $f_y'^*$	–3.521	–4.509	–4.471	–4.466
	New z'^*	–1.765	–2.479	–2.558	–2.567
Step 7	New β	3.939	5.145	5.151	5.151
	$\Delta\beta$		1.206	0.006	0.0001
Step 8	New f_y^*	24.464	24.112	24.207	24.22
	New z^*	49.235	47.307	47.093	47.07
	New $g(\,)$	64.500	0.679	–0.020	–0.0002

Convergence criteria in Steps 7 and 8: (1) $|\Delta\beta| \le 0.001$, (2) $|g(\,)| \le 0.001$.
The final checking point is (24.22, 47.07).

- Step 5. Equation 7.77 is used to find the partial derivatives in the equivalent normal space as

$$\left(\frac{\partial g}{\partial F_y'}\right)^* = \left(\frac{\partial g}{\partial F_y}\right)^* \sigma_{F_y}^N = 54 \times 3.79 = 204.69$$

$$\left(\frac{\partial g}{\partial Z'}\right)^* = \left(\frac{\partial g}{\partial Z}\right)^* \sigma_Z^N = 38 \times 2.7 = 102.60.$$

- Step 6. The coordinates of the new design point in the equivalent standard normal space are computed using the recursive formula of Equation 7.72 as

$$\text{New} \begin{Bmatrix} f_y'^* \\ z'^* \end{Bmatrix} = \frac{1}{\left[\left\{\left(\frac{\partial g}{\partial F_y'}\right)^*\right\}^2 + \left\{\left(\frac{\partial g}{\partial Z'}\right)^*\right\}^2\right]} \left[\left(\frac{\partial g}{\partial F_y'}\right)^* f_y'^* + \left(\frac{\partial g}{\partial Z'}\right)^* z'^* - g(\,)\right] \begin{Bmatrix} \left(\frac{\partial g}{\partial F_y'}\right)^* \\ \left(\frac{\partial g}{\partial Z'}\right)^* \end{Bmatrix}$$

$$= \frac{1}{[204.69^2 + 102.60^2]}(204.69 \times 0.05 + 102.60 \times 0.0 - 912)\begin{Bmatrix}204.69\\102.60\end{Bmatrix}$$

$$= \begin{Bmatrix}-3.521\\-1.765\end{Bmatrix}.$$

- Step 7. Using Equation 7.79, we find the value of β to be

$$\beta = \sqrt{(-3.521)^2 + (-1.765)^2} = 3.939.$$

The check for convergence at this step will start during the second iteration. In the second iteration, β is calculated as 5.145. Therefore, the change in the value of β between the first and second iterations is $1.206 > 0.001$.

- Step 8. Equation 7.80 is used to find the coordinates of the new iteration point in the original space:

$$f_y^* = \mu_{F_y}^N + \sigma_{F_y}^N f_y'^* = 37.81 + 3.79 \times (-3.521) = 24.464$$

$$z^* = \mu_Z^N + \sigma_Z^N z'^* = 54.0 + 2.7 \times (-1.765) = 49.235.$$

At these values, the performance function is evaluated as

$$g(\,) = 24.464 \times 49.235 - 1.140 = 64.5.$$

The convergence criterion for $g(\,)$ is checked. The current $g(\,)$ value is greater than the tolerance level of 0.001. Therefore, proceed to the next iteration at Step 3.

The search is stopped after four iterations since the value of β has converged to 5.151 and the value of $g(\,)$ has become less than 0.001. As expected, both FORM methods gave identical results. However, the advantage of FORM Method 2 is clear; it does not require solution of the limit state equation and simply uses a recursive formula to converge to the design point. Comparing Tables 7.2 and 7.3, we can observe that the same quantities are computed in both methods. The only difference is in how the new iteration point is computed. Also, during the first iteration, since both the algorithms were started from the mean values of the variables, many of the quantities have the same values.

Observations

Several important observations can be made at this time. The safety index obtained using any one of the FORM methods is different than the safety indexes shown in Table 7.1. It is also interesting to note that if the stress formulation of the limit state was considered, the safety index would have the same value (i.e., 5.151) as the strength formulation. The Hasofer–Lind algorithm ignores the information on the distributions of the random variables, essentially assuming both random variables are normal; if it is used, β_{HL} is found to be 4.261.

The FORM methods clearly demonstrate that information on the distribution of random variables is important in calculating the safety index and the corresponding probability of failure. To amplify the point, the safety indices of the same beam problem just considered are calculated assuming F_y and Z have different distributions, the results are summarized in Table 7.4.

In this example, only two random variables are present in the limit state equation. However, there could be any number of random variables in the limit state equation; as

Table 7.4 Safety Index Under Various Probability Distributions

Limit state equation = $g(\ ) = F_y Z - 1{,}140 = 0$			
Random variables	F_y	Z	β
Probability Distribution	Normal	Normal	4.261
	Normal	Lognormal	4.266
	Lognormal	Normal	5.151
	Lognormal	Lognormal	5.213

long as they are uncorrelated, either of the two algorithms discussed here can be used without modification to calculate the safety index or the corresponding probability of failure. If the random variables are correlated, some modifications in the algorithm are necessary, as discussed in Chapter 8.

To demonstrate the application of FORM to a more complicated problem involving several random variables, the moment capacity of a singly reinforced rectangular prismatic concrete beam is considered here. The moment capacity or resistance M_R of such a beam can be calculated using the following expression:

$$M_R = A_s f_y d\left(1 - \eta \frac{A_s}{bd} \frac{f_y}{f_c'}\right) \tag{7.81}$$

where A_s is the area of the tension reinforcing bars, f_y is the yield stress of the reinforcing bars, d is the distance from the extreme compression fiber to the centroid of the tension reinforcing bars, η is the concrete stress block parameter, f_c' is the compressive strength of concrete, and b is the width of the compression face of the member. It is extensively reported in the literature that all these variables are random. Their mean values and coefficients of variation are tabulated in Table 7.5. Assume further that the beam is subjected to a moment M, which is also a random variable. Its mean value and coefficient of variation are shown in Table 7.5.

The limit state equation for the problem can be expressed as

$$g(\ ) = A_s f_y d\left(1 - \eta \frac{A_s}{bd} \frac{f_y}{f_c'}\right) - M = 0. \tag{7.82}$$

Table 7.5 Uncertainty in the Design Parameters of a Reinforced Concrete Beam

Random variables	Mean	Coefficient of variation
A_s (in.^{2})	1.56	0.036
f_y (ksi)	47.7	0.15
f_c' (ksi)	3.5	0.21
b (in.)	8.0	0.045
d (in.)	13.2	0.086
η	0.59	0.05
M (kip–in.)	326.25	0.17

Table 7.6 Safety Index of a Reinforced Concrete Beam Under Various Probability Distributions

Random variables	Probability distribution			
A_s	Normal	Normal	Lognormal	Lognormal
f_y	Normal	Normal	Lognormal	Lognormal
f_c'	Normal	Normal	Lognormal	Lognormal
b	Normal	Normal	Lognormal	Lognormal
d	Normal	Normal	Lognormal	Lognormal
η	Normal	Normal	Lognormal	Lognormal
M	Normal	Lognormal	Normal	Lognormal
β	3.833	3.761	4.388	4.091

For various distributions of the random variables in Equation 7.82, the safety indices are calculated using the FORM method. It is not possible to show the detailed calculations in tabular form as in the previous example. A computer program is used for this purpose. The results summarized in Table 7.6 clearly indicate that the distributions of random variables play a very important role in safety index or probability of failure estimation.

7.6.3.5 Three-Parameter Equivalent Normal Transformation

Chen and Lind (1983) proposed an extension of the Rackwitz–Fiessler algorithm using a three-parameter approximation. It is known as the *Chen–Lind* method. For each nonnormal variable X_i, a third parameter A_i (referred to as a scale factor) is introduced, in addition to the mean and standard deviation, by imposing the condition that at the checking point the slopes of the probability density function must be equal for both the original and the equivalent normal distributions.

Wu and Wirsching (1987) proposed an alternative way to obtain a three-parameter equivalent normal transformation. In their approach, the scale factor A_i is approximately computed as the ratio of the failure probability estimate with the actual distribution to the failure probability estimate with the equivalent normal distribution (assuming a linearized g-function and replacing the effect of all other variables by a single normal variable). Then the other two parameters $\mu_{X_i}^N$ and $\sigma_{X_i}^N$ are computed by minimizing the sum of squares of the errors in the probability estimate between the actual distribution and the equivalent normal distribution.

7.7 RISK-BASED DESIGN FORMAT USING FORM

The concept of risk-based design format was introduced in Section 7.4. Considering performance functions of the form given by Equations 7.3 (for normal variables) and 7.20a (for lognormal variables), we estimated the central and nominal load and resistance factors in a design. Essentially, these factors were estimated using the MVFOSM method. Now that a more advanced reliability method, namely FORM, is known, it will be of interest to calculate the central and nominal load and resistance factors using this method. The following simple example for normal variables is given to demonstrate the procedure. However, with the same procedures, these factors can also be calculated for nonnormal and correlated variables (to be discussed in Chapter 8).

EXAMPLE 7.8

Example 7.1 given in Section 7.4.2 is considered again. A simply supported steel beam of 30-foot span is subjected to nominal dead and live loads of 70 and 100 psf, respectively. Assuming the beams are spaced 10 feet apart and are continuously laterally supported by the concrete slab, and using AISC's LRFD design criteria, an engineer suggests a W14 × 61 section of A36 steel. Denoting μ_D, μ_L, and μ_R as the mean values of the dead and live loads and the resistance of the W section, and the corresponding nominal values as D_N, L_N, and R_N, respectively, Ellingwood et al. (1980) showed that it may be very reasonable to assume $D_N/\mu_D = 1.05$, $L_N/\mu_L = 1.4$, and $R_N/\mu_R = 0.9$. The uncertainty in the dead load and live load and the resistance in terms of COV are 0.13, 0.37, and 0.13, respectively. It is necessary first to design the beam and then calculate the central and nominal load and resistance factors for this design corresponding to a reliability index β of 3.0.

SOLUTION

The design requirements using the LRFD concept can be expressed as

$$\phi R_N \geq \sum_{i=1}^{m} \gamma_i S_{N_i} \tag{7.83}$$

where ϕ is the resistance factor, subscript N denotes the nominal design values for load and resistance, γ_i is the ith load factor, and m is the number of loads that need to be considered for the critical load combination. For dead and live loads, Equation 7.83 becomes

$$\phi R_N = \gamma_D D_N + \gamma_L L_N. \tag{7.84}$$

The performance function for this design is

$$g(\,) = R - D - L. \tag{7.85}$$

From the information given in the problem, the mean values of dead and live loads are 70/1.05 = 66.67 psf and 100/1.4 = 71.43 psf, and the corresponding COVs are 0.13 and 0.37, respectively. The mean value of the moment caused by the applied dead load can be calculated as

$$\mu_{M_D} = \frac{66.67 \times 10 \times 30^2 \times 12}{8 \times 1000} = 900.05 \text{ k-in.}$$

The standard deviation of the bending moment due to the dead load M_D is

$$\sigma_{M_D} = 0.13 \times 900.05 = 117.01 \text{ kip-in.}$$

Similarly, the mean value of the moment caused by the applied live load is

$$\mu_{M_L} = \frac{71.43 \times 10 \times 30^2 \times 12}{8 \times 1{,}000} = 964.31 \text{ kip-in.}$$

The corresponding standard deviation of the live load moment M_L is

$$\sigma_{M_L} = 0.37 \times 964.31 = 356.79 \text{ kip-in.}$$

Equation 7.67 can be used to find the checking point for R, D, and L for $\beta = 3$:

$$r^* = \mu_R - \alpha_R \times 3 \times (0.13\mu_R)$$

$$d^* = 900.05 - \alpha_D \times 3 \times 117.01$$

and

$$l^* = 964.31 - \alpha_L \times 3 \times 356.79.$$

The checking point must satisfy the performance function given by Equation 7.85, that is,

$$g(\,) = 0 = (\mu_R - \alpha_R \times \beta \times \sigma_R) - (\mu_D - \alpha_D \times \beta \times \sigma_D) - (\mu_L - \alpha_L \times \beta \times \sigma_L). \tag{7.86}$$

The partial derivatives of the performance function with respect to R, D, and L evaluated at the checking point are

$$\frac{\partial g(\,)}{\partial R} = 1, \quad \frac{\partial g(\,)}{\partial D} = -1, \quad \frac{\partial g(\,)}{\partial L} = -1.$$

Equation 7.66 is used to find the corresponding direction cosines:

$$\alpha_R = \frac{\sigma_R}{\sqrt{\sigma_R^2 + \sigma_D^2 + \sigma_L^2}}, \quad \alpha_D = \frac{-\sigma_D}{\sqrt{\sigma_R^2 + \sigma_D^2 + \sigma_L^2}}, \quad \alpha_L = \frac{-\sigma_L}{\sqrt{\sigma_R^2 + \sigma_D^2 + \sigma_L^2}}.$$

Substituting these direction cosine values in Equation 7.86 and simplifying will result in

$$\frac{\mu_R - \mu_D - \mu_L}{\sqrt{\sigma_R^2 + \sigma_D^2 + \sigma_L^2}} = \beta. \tag{7.87}$$

For the problem under consideration, Equation 7.87 becomes

$$\frac{\mu_R - 900.05 - 964.31}{\sqrt{(0.13 \times \mu_R)^2 + 117.01^2 + 356.79^2}} = 3.$$

This is a quadratic equation in terms of the mean value of R. Solving the equation gives $\mu_R = 3{,}692.75$ kip-in. Thus, $\sigma_R = 0.13 \times 3{,}692.75 = 480.06$ kip-in. The direction cosines can now be evaluated as

$$\alpha_R = \frac{480.06}{\sqrt{480.06^2 + 117.01^2 + 356.79^2}} = \frac{480.06}{609.47} = 0.788$$

$$\alpha_D = \frac{-117.01}{609.47} = -0.192$$

and

$$\alpha_L = \frac{-356.79}{609.47} = -0.585.$$

For a linear performance function no iteration is necessary. The nominal value of the resisting bending moment is

$$R_N = 0.9 \times 3{,}692.75 = 3{,}323.48 \text{ kip-in.}$$

Since the unbraced length of the beam is zero, and we assume the section is compact, it is expected to develop full plastic moment. Thus, if A36 steel is used, the required plastic section modulus for the beam is

$$Z_{\text{req}} = \frac{3{,}323.48}{36} = 92.32 \text{ in.}^3$$

A W14×61 section of A36 steel is used for the beam. The same section can be obtained by designing the beam according to the AISC's LRFD code.

Using either Equation 7.56 or 7.67, for the ith normal variable, we can show that

$$x_i^* = \mu_{X_i} - \alpha_{X_i}\beta\sigma_{X_i} = \mu_{X_i}(1 - \alpha_{X_i}\beta\delta_{X_i}). \tag{7.88}$$

Thus, the central resistance and load factors are $(1 - \alpha_{X_i}\beta\delta_{X_i})$. For the problem under consideration, these factors are

$$\mu_\Phi = 1 - 0.788 \times 3 \times 0.13 = 0.693$$

$$\mu_{\gamma_D} = 1 + 0.192 \times 3 \times 0.13 = 1.075$$

$$\mu_{\gamma_L} = 1 + 0.585 \times 3 \times 0.37 = 1.649.$$

Thus, the design requires

$$0.693\mu_R \geq 1.075 \times \mu_D + 1.649 \times \mu_L. \tag{7.89}$$

Since the ratios of nominal to mean values are known for all the three parameters, Equation 7.89 can be rewritten in terms of nominal values as

$$0.693\left(\frac{R_N}{0.9}\right) \geq 1.075\left(\frac{D_N}{1.05}\right) + 1.649\left(\frac{L_N}{1.4}\right)$$

or

$$0.77R_N \geq 1.02D_N + 1.18L_N. \tag{7.90}$$

These resistance and load factors correspond to the reliability index β of 3. If the reliability index β is selected to be 4, indicating a much safer or conservative design, and the procedures just discussed are followed, the design requirements (similar to Equation 7.90) become

$$0.62R_N \geq 1.03D_N + 1.24L_N. \tag{7.91}$$

For this relatively more conservative design, the resistance factor decreased and load factors increased, as expected. For a specific application, the reliability index needs to be calibrated considering the acceptable practice in the profession. This simple example clearly demonstrates the mathematical basis for the resistance and load factors commonly used in design codes.

7.8 CONCLUDING REMARKS

The risk-based design concept is presented in this chapter. It was shown that the conventional safety factor–based deterministic designs in terms of capacity reduction factor and load factors, and the risk or probability-based load and resistance factor designs, are essentially parallel to each other. However, the risk-based design explicitly incorporates more information in developing these factors. It is perhaps a better and more comprehensive approach, and it empowers engineers to make better design decisions on a case-by-case basis. Using the fundamental concept of risk-based design, various reliability analysis methods with different degrees of complexity and completeness are available. They are presented systematically in this chapter. The steps to extract the necessary information are identified and clarified with the help of examples.

The information presented in the first seven chapters of this book is expected to provide readers with the necessary background to calculate the probability of failure or the reliability of simple structural components. Some of the more advanced topics on risk and reliability estimation are presented in Chapter 8.

7.9 PROBLEMS

7.1 A simply supported steel beam with a span of 40 ft needs to be designed to carry a dead load of 80 psf and a live load of 50 psf. The beams are spaced 10 ft apart and are laterally supported at the supports and at the midspan. Using A36 steel and the AISC's LRFD code, a steel section of W21 × 73 is suggested. Assume the following:

Nominal moment capacity of the beam, $M_R = 151 \times 36 = 5{,}436$ k-in.

Nominal M_R = mean − 2 × standard deviation value

Nominal dead load = 80 psf = mean + 1 × standard deviation value

Nominal live load = 50 psf = mean + 2 × standard deviation value

The COVs of M_R, dead load, and live load are 0.13, 0.13, and 0.37, respectively. Consider all the random variables to be normally distributed. Using the concept discussed in Section 7.4, calculate the following:

(a) The probability of failure of the beam.

(b) The corresponding central safety factor, capacity reduction factor, and load factors.

(c) The nominal safety factor, capacity reduction factor, and load factors.

7.2 If all the random variables in Problem 7.1 are lognormally distributed, calculate the following:

(a) The probability of failure of the beam.

(b) The corresponding central safety factor, capacity reduction factor, and load factors.

(c) The nominal safety factor, capacity reduction factor, and load factors.

7.3 The bearing capacity, C, of soil under a square foundation of size 9 ft^2 is determined to be a random variable with a mean of 3 ksf and a standard deviation of 0.5 ksf. The applied axial load, P, acting on the foundation is also a random variable with a mean of 15 kip and a standard deviation of 2 kip. Assume C and P are statistically independent and no information on their distribution is available. Using a limit state function of the form $g(\ ) = 9C - P$ and the MVFOSM method, calculate the reliability index for the foundation.

7.4 Consider Problem 7.3.

(a) If C and P are statistically independent normal random variables with the same means and standard deviations, calculate the reliability index and the corresponding probability of failure of the foundation.

(b) If C and P are statistically independent lognormal random variables with the same means and standard deviations, will the reliability index be different? Can the probability of failure of the foundation be calculated exactly?

7.5 A simply supported beam of span $L = 360$ inches is loaded by a uniformly distributed load w in kip/in. and a concentrated load P in kip applied at the midspan. The maximum deflection of the beam at the midspan can be calculated as

$$\delta_{\max} = \frac{5}{384}\frac{wL^4}{EI} + \frac{1}{48}\frac{PL^3}{EI}.$$

A beam with $EI = 63.51 \times 10^6$ kip-in.^{2} is selected to carry the load. Both w and P are statistically independent random variables with mean values estimated to be 0.2 kip/in. and 25 kip, respectively. The corresponding standard deviations are 0.03 kip/in. and 2.5 kip, respectively. Assume the distributions of w and P are unavailable. The allowable deflection, δ_a, for the beam is a constant of value 1.5 inch. Considering the limit state equation of the form $g(\ ) = \delta_a - \delta_{\max} = 0$, calculate the reliability index for the beam in deflection using the MVFOSM method.

7.6 In Problem 7.5, suppose both w and P are statistically independent normal random variables with the same means and standard deviations. Using FORM Method 1, calculate the reliability index for the beam. Discuss why the reliability indexes obtained in Problems 7.5 and 7.6 are the same.

7.7 Consider Problem 7.6. Calculate the reliability index of the beam using FORM Method 2. Is it identical to the value obtained in Problem 7.6?

7.8 In Problem 7.5, suppose w is a normal random variable and P is a lognormal random variable with the same means and standard deviations. If w and P are statistically independent, calculate the reliability index for the beam using FORM Method 1. Discuss why the reliability indices obtained in Problems 7.6 and 7.8 are different.

7.9 Consider Example 7.6. The limit state equation according to the strength formulation is $g(\ ) = F_y Z - 1{,}140 = 0$, in which F_y has a lognormal distribution with a mean and COV of 38 ksi and 0.1, respectively, and Z has a normal distribution with a mean and COV of 54 in.^{3} and 0.05, respectively. Table 7.2 indicates that for this problem, $\beta = 5.151$ and the checking point is (24.22, 47.07). The limit state equation is rewritten in the stress formulation as

$$g(\ ) = F_y - \frac{1{,}140}{Z} = 0.$$

Using either Method 1 or Method 2, show that the reliability index and the checking point will be unchanged according to FORM.

7.10 The fully plastic flexural capacity of a steel beam section can be estimated as $Y Z$, where Y = the yield strength of steel and Z = the plastic section modulus of the section. If the applied bending moment at a location of interest is M, the performance function may be defined as

$$g(\ ) = YZ - M.$$

Assume Z is a constant of value 50 in.^{3} and Y and M are independent normal random variables with mean values of 40 ksi and 1,000 kip-in., respectively, and corresponding COVs of 0.125 and 0.20, respectively. Estimate the reliability of the beam using FORM (Method 1 or Method 2).

7.11 In Problem 7.10, if Y and M are independent lognormal random variables with the same means and COVs, and Z is a constant of value 50 in.^{3}, estimate the reliability of the beam using FORM (Method 1 or Method 2).

7.12 In Problem 7.10, consider Y to be a normal and M to be a lognormal random variable with the same means and COVs, and Z to be a constant with a value of 50 in.^{3} Estimate the reliability of the beam using FORM (Method 1 or Method 2).

7.13 A simply supported steel beam needs to be designed to carry dead, live, and snow loads. The nominal bending moments produced by these loads are 945, 1,350, and 675 kip-in., respectively. The nominal to mean value ratios for the dead, live, and snow loads and resistance are assumed to be 1.05, 1.4, 1.2, and 0.9, respectively. The uncertainties in the dead, live, and snow loads and the resistance in terms of COV are 0.13, 0.37, 0.26, and 0.13, respectively. Assume they are normal random variables.

(a) If the yield stress F_y of the steel section being used is 36 ksi, and assuming that the beam will develop plastic moment M_p(i.e., $M_p = F_y Z$, where Z is the plastic section modulus) design the beam in terms of Z and select the corresponding W section.

(b) Calculate the central and nominal load and resistance factors for this design corresponding to a reliability index β of 3.0.

7.14 In Problem 7.13, suppose the beam needs to be designed for the dead and snow loads only. The nominal bending moments due to dead and snow loads and their COV and the resistance are the same as in Problem 7.13.

(a) For the yield stress of steel of 36 ksi, design the beam.

(b) Calculate the central and nominal load and resistance factors for this design corresponding to a reliability index of 2.5.

Chapter 8

Advanced Topics on Reliability Analysis

8.1 INTRODUCTORY COMMENTS

The essential concepts in calculating the reliability of a component are discussed in Chapter 7. As mentioned earlier, the knowledge in this area of study is still evolving. The discussion would be incomplete if some of the more advanced topics on risk and reliability were not mentioned and briefly discussed. These areas will be of considerable interest to serious students who would like to advance the state of the art. Some of these more advanced topics are presented in this chapter.

8.2 SECOND-ORDER RELIABILITY METHODS (SORM)

As mentioned in Chapter 7, limit states (explicit or implicit, linear or nonlinear), are essential in risk and reliability analysis. The computations required for the reliability analysis of problems with linear limit state equations are relatively simple. However, the limit state could be nonlinear either due to the nonlinear relationship between the random variables in the limit state equation or due to some variables being nonnormal. A linear limit state in the original space becomes nonlinear when transformed to the standard normal space (which is where the search for the minimum distance point is conducted) if any of the variables is nonnormal. Also, the transformation from correlated to uncorrelated variables might induce nonlinearity; this transformation is discussed in detail in Section 8.3. If the joint probability density function, PDF, of the random variables decays rapidly as one moves away from the minimum distance point, then the first-order estimate of failure probability is quite accurate. If the decay of the joint PDF is slow and the limit state is highly nonlinear, then one has to use a higher-order approximation for the failure probability computation. Consider the two limit

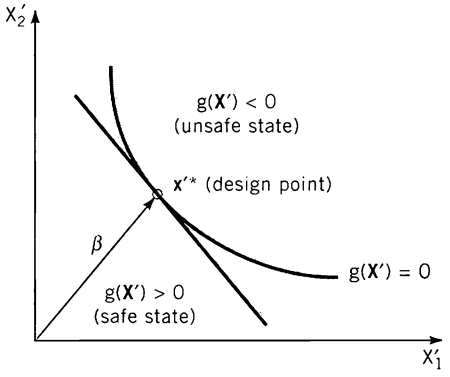


Figure 8.1 Linear and Nonlinear Limit States

states shown in Figure 8.1, one linear and one nonlinear. Both limit states have the same minimum distance point, but the failure domains, shown by the shaded regions, are different for the two cases. The FORM approach will give the same reliability estimate for both cases. But it is apparent that the failure probability of the nonlinear limit state should be less than that of the linear limit state, due to the difference in the failure domains. The curvature of the nonlinear limit state is ignored in the FORM approach, which uses only a first-order approximation at the minimum distance point. Thus the curvature of the limit state around the minimum distance point determines the accuracy of the first-order approximation in FORM. The curvature of any equation is related to the second-order derivatives with respect to the basic variables. Thus, the second-order reliability method (SORM) improves the FORM result by including additional information about the curvature of the limit state.

The Taylor series expansion of a general nonlinear function $g(X_1, X_2, ..., X_n)$ at the value $(x_1^*, x_2^*, \ldots, x_n^*)$ is

$$g(X_1, X_2, \ldots, X_n) = g(x_1^*, x_2^*, \ldots, x_n^*) + \sum_{i=1}^{n} (x_i - x_i^*) \frac{\partial g}{\partial X_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_i^*)(x_j - x_j^*) \frac{\partial^2 g}{\partial X_i \partial X_j} + \ldots \tag{8.1}$$

where the derivatives are evaluated at the design point of the X_i's.

The variables $(X_1, X_2, ..., X_n)$ are used in Equation 8.1 in a generic sense. One should use the appropriate set of variables and notation depending on the space being considered. In the case of reliability analysis, the second-order approximation to $g(\ )$ is being constructed in the space of standard normal variables, at the minimum distance point. The following notation is used in this section: X_i refers to a random variable in the original space, and Y_i refers to the random variable in the equivalent uncorrelated standard normal space. If all the variables are uncorrelated, $Y_i = (X_i - \mu_{X_i}^N)/\sigma_{X_i}^N$ where $\mu_{X_i}^N$ and $\sigma_{X_i}^N$ are the equivalent normal mean and standard deviation of X_i at the design point x_i^*. The transformation from X_i to Y_i for correlated variables is discussed in Section 8.3.

In the Taylor series approximation given in Equation 8.1, FORM ignores the terms beyond the first-order term (involving first-order derivatives), and SORM ignores the terms beyond the second-order term (involving second-order derivatives).

The SORM approach was first explored by Fiessler et al. (1979) using various quadratic approximations. A simple closed-form solution for the probability computation using a second-order approximation, p_{f_2}, was given by Breitung (1984) using the theory of asymptotic approximations as

$$p_{f_2} \approx \Phi(-\beta)\prod_{i=1}^{n-1}(1+\beta\kappa_i)^{-1/2} \tag{8.2}$$

where κ_i denotes the principal curvatures of the limit state at the minimum distance point, and β is the reliability index using FORM. Breitung showed that this second-order probability estimate asymptotically approaches the first-order estimate as β approaches infinity, if $\beta\kappa_i$ remains constant. Refer to Hohenbichler et al. (1987) for a theoretical explanation of FORM and SORM using the concept of asymptotic approximations.

In Equation 8.2, it is necessary to compute the principal curvatures κ_i. To do this, first the Y_i variables (in the **Y** space) are rotated to another set of variables, denoted as Y_i', such that the last Y_i' variable coincides with the vector $\boldsymbol{\alpha}$, the unit gradient vector of the limit state at the minimum distance point. This is shown in Figure 8.2 for a problem with two random variables. It is apparent that this is simply a rotation of coordinates.

The transformation from the **Y** space to the **Y′** space is an orthogonal transformation:

$$\mathbf{Y}' = \mathbf{R}\mathbf{Y} \tag{8.3}$$

where **R** is the rotation matrix. For the simple case of two random variables, it is

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{8.4}$$

where θ is the angle of rotation as shown in Figure 8.2 (counterclockwise rotation of the axes gives positive θ). When the number of variables is more than two, the **R** matrix is computed in two steps. In Step 1, first a matrix, $\mathbf{R}_0$, is constructed as follows:

$$\mathbf{R}_0 = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_1 & \alpha_2 & \cdot & \cdot & \cdot & \alpha_n \end{bmatrix} \tag{8.5}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the direction cosines, that is, components of the unit gradient vector $\boldsymbol{\alpha}$ shown in Figure 8.2. In Step 2, a Gram–Schmidt orthogonalization procedure (refer to Appendix 6) is applied to this matrix, and the resulting matrix is **R**.

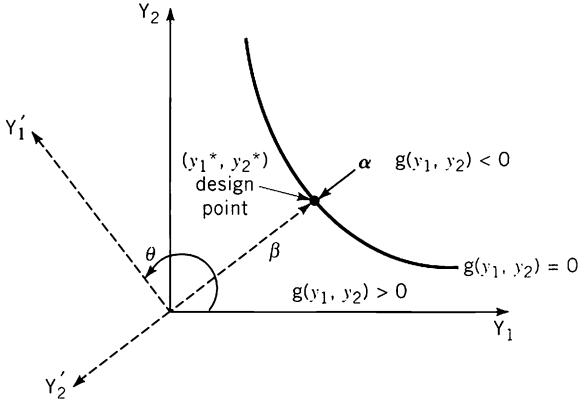


Figure 8.2 Rotation of Coordinates

Once the **R** matrix is obtained, a matrix **A,** whose elements are denoted as a_{ij}, is computed as

$$a_{ij} = \frac{(\mathbf{RDR}^t)_{ij}}{|\nabla G(\mathbf{y}^*)|}, \quad i, j = 1, 2, \ldots, n-1 \tag{8.6}$$

where **D** is the $n \times n$ second-derivative matrix of the limit state surface in the standard normal space evaluated at the design point, **R** is the rotation matrix, and $|\nabla G(\mathbf{y}^*)|$ is the length of the gradient vector in the standard normal space.

In the rotated space, the last variable Y_n coincides with the β-vector computed in FORM. In the next step, the last row and last column in the **A** matrix and the last row in the $\mathbf{Y}'$ vector are dropped to take this factor into account. The limit state can be rewritten in terms of a second-order approximation in this rotated standard normal space $\mathbf{Y}'$ as

$$y'_n = \beta + \frac{1}{2}\mathbf{y}'^t\mathbf{A}\mathbf{y}' \tag{8.7}$$

where the matrix **A** is now of the size $(n-1) \times (n-1)$.

Finally, the main curvatures κ_i, used in Breitung's formula, are computed as the eigenvalues of the matrix **A**. Once the κ_i's are computed, Breitung's formula can be used to compute the second-order estimate of the probability of failure.

Breitung's SORM method uses a parabolic approximation; that is, it does not use a general second-order approximation. (It ignores the mixed terms and their derivatives in the Taylor series approximation in Equation 8.1.) Also, as mentioned earlier, it uses the theory of asymptotic approximation to derive the probability estimate. The asymptotic formula is accurate only for large values of β, which is the case for practical high-reliability problems. However, if the value of β is low, the SORM estimate could be inaccurate. Tvedt (1990) developed two alternative SORM formulations to take care of these problems. Tvedt's method uses a parabolic and a general second-order approximation to the limit state, and it does not use asymptotic approximations. Refer to Tvedt (1990) for a detailed presentation of this method.

Der Kiureghian et al. (1987) approximated the limit state by two semiparabolas using curve-fitting at several discrete points around the design point and used both sets of curvature in Breitung's formula (Equation 8.2). This strategy helps to avoid the computation of a full second-derivative matrix using the original limit state and is efficient for problems with a large number of random variables.

EXAMPLE 8.1

Example 7.6 in Section 7.6.3 of Chapter 7, discussed in detail in Table 7.2, is considered again. Using FORM and the strength formulation, assuming F_y to be a lognormal variable with a mean of 38 ksi and standard deviation of 3.8 ksi, and assuming Z to be a normal variable with a mean of 54 in.^{3} and standard deviation of 2.7 in.^{3}, we found the safety index to be 5.151. Using SORM, estimate the safety index.

SOLUTION

The estimation of a safety index using FORM and to a greater extent using SORM is rarely undertaken using hand calculations; computer programs are used for this purpose. However, as in FORM where the detailed calculations are summarized in Table

7.2, in this section the detailed steps in SORM are explained for better understanding of the concept.

Table 7.2 reveals that the final design point in the original variable space is (24.22 ksi, 47.07 in.^{3}), and the corresponding direction cosines for F_y and Z are 0.867 and 0.498, respectively. The equivalent normal mean and standard deviation of F_y at the design point are 35.008 and 2.416, respectively. Transforming F_y and Z from the original to standard normal space results in

$$Y_{F_y} = \frac{F_y - \mu^N_{F_y}}{\sigma^N_{F_y}}$$

and

$$Y_Z = \frac{Z - \mu_Z}{\sigma_Z}.$$

The coordinates of the design point in the standard normal space become

$$y^*_{F_y} = \frac{24.22 - 35.008}{2.416} = -4.466$$

and

$$y^*_Z = \frac{47.07 - 54.00}{2.7} = -2.567.$$

The design point is graphically shown in the standard normal space in Figure 8.3.

The first step in SORM is to construct the rotation matrix $\mathbf{R}$ in Equation 8.4 or 8.5. For the two-variable problem under consideration, Equation 8.4 is sufficient. In this example, Y_{F_y} is the first coordinate and Y_Z is the second coordinate. In the rotated coordinates, the second coordinate Y_Z' needs to coincide with the unit gradient vector $\boldsymbol{\alpha}$. The corresponding rotation angle θ is shown in Figure 8.3. Therefore, the $\mathbf{R}$ matrix is

$$\mathbf{R} = \begin{bmatrix} 0.498 & -0.867 \\ 0.867 & 0.498 \end{bmatrix}.$$

Notice that the elements of $\mathbf{R}$ are easily available from the direction cosines, that is, the components of the unit gradient vector $\boldsymbol{\alpha}$.

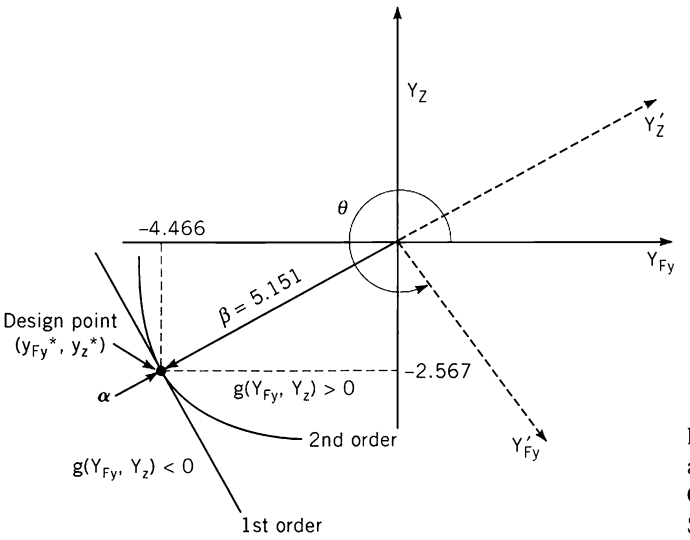


Figure 8.3 Design Point and Rotation of Coordinates in the Standard Normal Space

The next step is to construct the **D** matrix, containing second derivatives of the performance function, in the standard normal space. For the performance function of $g(\,) = F_y\,Z - 1{,}140$ in the original space, using the chain rule of differentiation, the elements of **D** are

$$\frac{\partial^2 g(\,)}{\partial F_y'^2} = \frac{\partial}{\partial F_y}\left\{\left[\frac{\partial g(\,)}{\partial F_y}\frac{\partial F_y}{\partial F_y'}\right]\right\}\frac{\partial F_y}{\partial F_y'} = \frac{\partial}{\partial F_y}\left[Z\sigma_{F_y}^N\right]\sigma_{F_y}^N = 0$$

$$\frac{\partial^2 g(\,)}{\partial Z'^2} = \frac{\partial}{\partial Z}\left\{\left[\frac{\partial g(\,)}{\partial Z}\frac{\partial Z}{\partial Z'}\right]\right\}\frac{\partial Z}{\partial Z'} = \frac{\partial}{\partial Z}\left[F_y\sigma_Z\right]\sigma_Z = 0$$

$$\frac{\partial^2 g(\,)}{\partial F_y' \partial Z'} = \frac{\partial}{\partial F_y}\left\{\left[\frac{\partial g(\,)}{\partial Z}\frac{\partial Z}{\partial Z'}\right]\right\}\frac{\partial F_y}{\partial F_y'} = \frac{\partial}{\partial F_y}\left[F_y\sigma_Z\right]\sigma_{F_y}^N = \sigma_Z\sigma_{F_y}^N.$$

Therefore, matrix **D** is assembled as

$$\mathbf{D} = \begin{bmatrix} 0 & \sigma_Z\sigma_{F_y}^N \\ \sigma_Z\sigma_{F_y}^N & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2.7\times 2.416 \\ 2.7\times 2.416 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6.523 \\ 6.523 & 0 \end{bmatrix}.$$

Next, matrix **A** from Equation 8.6 needs to be computed. To do this, the length of the gradient vector in the standard normal space at the design point $|\nabla G(\mathbf{y}^*)|$ is needed. Normally, this would be readily available from the FORM analysis. In fact, the direction cosines given previously are simply the components of the unit gradient vector. However, the computation is shown in detail as follows, for the sake of clarity.

In order to evaluate $\nabla G(\mathbf{y}^*)$ in Equation 8.6, the following two partial derivatives need to be evaluated:

$$\frac{\partial g(\,)}{\partial F_y'} = \frac{\partial g(\,)}{\partial F_y}\frac{\partial F_y}{\partial F_y'} = Z\sigma_{F_y}^N$$

and

$$\frac{\partial g(\,)}{\partial Z'} = \frac{\partial g(\,)}{\partial Z}\frac{\partial Z}{\partial Z'} = F_y\sigma_Z.$$

At the design point, the two partial derivatives are

$$\nabla G(\mathbf{y}^*) = \begin{Bmatrix} z^*\times\sigma_{F_y} \\ f_y^*\times\sigma_Z \end{Bmatrix} = \begin{Bmatrix} 47.07\times 2.416 \\ 24.22\times 2.7 \end{Bmatrix} = \begin{Bmatrix} 113.721 \\ 65.394 \end{Bmatrix}.$$

The length of the vector is

$$|\nabla G(\mathbf{y}^*)| = \sqrt{(113.721)^2 + (65.394)^2} = 131.182.$$

Equation 8.6 is used to compute matrix **A** as

$$[\mathbf{A}] = \frac{1}{131.182}\begin{bmatrix} 0.498 & -0.867 \\ 0.867 & 0.498 \end{bmatrix}\begin{bmatrix} 0 & 6.523 \\ 6.523 & 0 \end{bmatrix}\begin{bmatrix} 0.498 & 0.867 \\ -0.867 & 0.498 \end{bmatrix}$$

$$= \begin{bmatrix} -0.043 & -0.025 \\ -0.025 & 0.043 \end{bmatrix}$$

As explained in the text, the rotation of coordinates makes the last variable coincide with the $\boldsymbol{\beta}$-vector. Therefore, the last row and the last column of **A** are dropped for future consideration. For this two-variable problem, that leaves the matrix **A** with just one element, $a_{11} = -0.043$. Therefore, the eigenvalue of this one-element matrix is simply $\kappa_1 = a_{11} = -0.043$.

Table 8.1 Comparison of FORM and SORM Results

$g(\ ) = F_y Z - 1{,}140 = 0,$

$\mu_{F_y} = 38$ ksi, $\delta_{F_y} = 0.1$, $\mu_Z = 54$ in.^{3}, and $\delta_Z = 0.05$.

Probability distributions		Safety indexes	
F_y	Z	FORM	SORM
Normal	Normal	4.261	4.246
Lognormal	Normal	5.151	5.139
Normal	Lognormal	4.266	4.259
Lognormal	Lognormal	5.213	5.211

Equation 8.2 is used to compute the probability of failure using the second-order approximation:

$$p_{f_2} \approx \Phi(-5.151)[1 + 5.151 \times (-0.043)]^{-1/2} = 1.4708 \times 10^{-7}.$$

For the sake of comparison with FORM, a new safety index is computed as the inverse of this failure probability estimate as

$$\beta_{\text{SORM}} = -\Phi^{-1}(1.4708 \times 10^{-7}) = 5.139.$$

Note that for this example, the second-order approximation should give a larger failure probability estimate, as shown in Figure 8.3. Correspondingly, the safety index for SORM is less than that for FORM.

Considering the same example and using various distributions of F_y and Z, we can calculate the safety indexes according to the FORM and SORM methods. The results are summarized in Table 8.1. The underlying distributions of random variables have a considerable amount of influence on the safety index calculations; however, their differences are not significant for the FORM and SORM methods in this problem, since the limit state is barely nonlinear.

8.3 RELIABILITY ANALYSIS WITH CORRELATED VARIABLES

The FORM and SORM methods described in the previous sections implicitly assume that the basic variables $X_1, X_2, \ldots, X_n$ are uncorrelated. However, usually some variables are correlated. Consider the X_i's in Equation 7.34 to be correlated variables with means μ_{X_i}, standard deviations σ_{X_i}, and the covariance matrix represented as

$$[\mathbf{C}] = \begin{bmatrix} \sigma_{X_1}^2 & \text{cov}(X_1, X_2) & \cdots & \text{cov}(X_1, X_n) \\ \text{cov}(X_2, X_1) & \sigma_{X_2}^2 & \cdots & \text{cov}(X_2, X_n) \\ \vdots & \vdots & & \vdots \\ \text{cov}(X_n, X_1) & \text{cov}(X_n, X_2) & \cdots & \sigma_{X_n}^2 \end{bmatrix}. \tag{8.8}$$

If the reduced variables X_i' are defined as

$$X_i' = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}, \quad (i = 1, 2, \ldots, n) \tag{8.9}$$

then it can be shown that the covariance matrix $[\mathbf{C}']$ of the reduced variables X'_i is

$$[\mathbf{C}'] = \begin{bmatrix} 1 & \rho_{X_1,X_2} & \cdots & \rho_{X_1,X_n} \\ \rho_{X_2,X_1} & 1 & \cdots & \rho_{X_2,X_n} \\ \vdots & \vdots & & \vdots \\ \rho_{X_n,X_1} & \rho_{X_n,X_2} & \cdots & 1 \end{bmatrix} \tag{8.10}$$

where ρ_{X_i,X_j} is the correlation coefficient of the X_i and X_j variables.

The FORM and SORM methods can be used if the X_i's are transformed into uncorrelated reduced $\mathbf{Y}$ variables and Equation 7.34 is expressed in terms of the $\mathbf{Y}$ variables. This can be done using the following equation:

$$\{\mathbf{X}\} = \left[\boldsymbol{\sigma}_{\mathbf{X}}^{N}\right][\mathbf{T}]\{\mathbf{Y}\} + \left\{\mu_{\mathbf{X}}^{N}\right\} \tag{8.11}$$

in which $\mu_{X_i}^{N}$ and $\sigma_{X_i}^{N}$ are the equivalent normal mean and standard deviation, respectively, of the X_i variables evaluated at the design point on the failure surface using Equations 7.59 and 7.61, and $\mathbf{T}$ is a transformation matrix to convert the correlated reduced $\mathbf{X}'$ variables to uncorrelated reduced $\mathbf{Y}$ variables. Note that the matrix containing the equivalent normal standard deviation in Equation 8.11 is a diagonal matrix. The $\mathbf{T}$ matrix can be shown to be

$$[\mathbf{T}] = \begin{bmatrix} \theta_1^{(1)} & \theta_1^{(2)} & \cdots & \theta_1^{(n)} \\ \theta_2^{(1)} & \theta_2^{(2)} & \cdots & \theta_2^{(n)} \\ \vdots & \vdots & & \vdots \\ \theta_n^{(1)} & \theta_n^{(2)} & \cdots & \theta_n^{(n)} \end{bmatrix}. \tag{8.12}$$

$[\mathbf{T}]$ is basically an orthogonal transformation matrix consisting of the eigenvectors of the correlation matrix $[\mathbf{C}']$ (Equation 8.10). $\{\boldsymbol{\theta}^{(i)}\}$ is the eigenvector of the ith mode. $\theta_1^{(i)}, \theta_2^{(i)}, \ldots, \theta_n^{(i)}$ are the components of the ith eigenvector.

Using Equation 8.11, we can write Equation 7.34 in terms of reduced uncorrelated normal $\mathbf{Y}$ variables. For this case, estimating the probability of structural failure is simple, as outlined in this section.

For practical large problems, the correlated variables may also be transformed into uncorrelated variables through an orthogonal transformation of the form

$$\mathbf{Y} = \mathbf{L}^{-1}(\mathbf{X}')' \tag{8.13}$$

where $\mathbf{L}$ is the lower triangular matrix obtained by Cholesky factorization of the correlation matrix $[\mathbf{C}']$. If the original variables are nonnormal, their correlation coefficients change on transformation to equivalent normal variables. Der Kiureghian and Liu (1985) developed semiempirical formulas for fast and reasonably accurate computation of $[\mathbf{C}']$.

The procedure discussed here can be applied when the marginal distributions of all the variables as well as the covariance matrix are known. When the joint distributions of all the correlated nonnormal variables are available, an equivalent set of independent normal variables can be obtained using the Rosenblatt transformation. From a practical point of view, this situation would be rare unless all the variables are either normal or lognormal. Furthermore, it is not possible to define the joint probability density function uniquely using the information on marginal distributions and the covariance matrix (Bickel and Doksum, 1977).

When the random variables are correlated, two types of problems can be envisioned. In the first type, all the random variables are normal; however, they are correlated to each other. In the second type, some or all the correlated random variables are nonnormal. These cases are explained with the help of examples in the following sections.

8.3.1 Correlated Normal Variables

EXAMPLE 8.2

Example 7.3 considered in Section 7.6.1 and represented by the limit state given by Equation 7.43, can be considered again. To illustrate the procedures for safety index or probability of failure evaluation for the correlated normal variables case, both random variables F_y and Z are considered to be normal with the mean and the coefficient of variation given in Table 8.1. The correlation coefficient between them is assumed to be 0.3.

Because the random variables are all normal, it is only necessary to rewrite the limit state Equation 7.43 in terms of the **Y** variables, that is, uncorrelated normal variables using Equation 8.11. The transformation matrix **T** needs to be evaluated at this stage using Equation 8.12. The correlation matrix $[\mathbf{C}']$ given by Equation 8.10 for the problem under consideration is

$$[\mathbf{C}'] = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix}.$$

The two eigenvalues of $[\mathbf{C}']$ can be calculated by solving the following equation

$$\det \begin{bmatrix} (1-\lambda) & 0.3 \\ 0.3 & (1-\lambda) \end{bmatrix} = 0$$

or

$$(1-\lambda)^2 - 0.3^2 = 0$$

or

$$\lambda_1 = 0.7, \quad \text{and} \quad \lambda_2 = 1.3.$$

The λ_i's are the variance of the Y_i's.

The corresponding eigenvectors can be obtained by solving the following equation:

$$\begin{bmatrix} (1-\lambda_i) & 0.3 \\ 0.3 & (1-\lambda_i) \end{bmatrix} \begin{Bmatrix} \theta_1^{(i)} \\ \theta_2^{(i)} \end{Bmatrix} = 0.$$

For each eigenvalue, the corresponding eigenvector can be calculated. For the problem under consideration, the eigenvectors are $\{\theta^{(1)}\} = \{1 \ -1\}$ and $\{\theta^{(2)}\} = \{1 \ 1\}$, respectively. The $[\mathbf{T}]$ matrix in Equation 8.12 represents the normalized eigenvectors and can be expressed as

$$[\mathbf{T}] = \begin{bmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{bmatrix}.$$

Equation 8.11 can now be expressed as

$$\begin{Bmatrix} F_y \\ Z \end{Bmatrix} = \begin{bmatrix} 3.8 & 0 \\ 0 & 2.7 \end{bmatrix} \begin{bmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} + \begin{Bmatrix} 38 \\ 54 \end{Bmatrix}$$

or

$$F_y = 2.687Y_1 + 2.687Y_2 + 38$$

and

$$Z = -1.909Y_2 + 1.909Y_2 + 54.$$

Equation 7.43 can now be rewritten in terms of the uncorrelated normal **Y** variables as

$$(2.6587Y_1 + 2.687Y_2 + 38)(-1.909Y_1 + 1.909Y_2 + 54) - 1{,}140 = 0$$

or

$$g(\,) = -5.129483\ Y_1^2 + 5.129483\ Y_2^2 + 72.556\ Y_1 + 217.640\ Y_2 + 912 = 0. \qquad (8.14)$$

Considering Equation 8.14 to be the limit state equation and using the nine steps of FORM Method 1, we can estimate the reliability index for the problem under consideration. All the necessary steps are summarized in Table 8.2. FORM Method 2 can also be used for this purpose, as discussed in Chapter 7.

Table 8.2 Steps in FORM Method 1 for Correlated Normal Variables

Step 1	$g(\ ) = -5.129483\ Y_1^2 + 5.129483\ Y_2^2 + 72.556\ Y_1 + 217.64\ Y_2 + 912 = 0$						
Step 2	β	5.0				3.927	
Step 3	y_1^*	0.0	−0.996	−1.477	−1.535	−1.212	−1.104
	y_2^*	0.0	−5.536	−5.336	−5.302	−4.160	−4.218
Step 4	$\mu_{Y_1}^N$	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma_{Y_1}^N$	$\sqrt{0.7}$ = 0.837	0.837	0.837	0.837	0.837	0.837
	$\mu_{Y_2}^N$	0.0	0.0	0.0	0.0	0.0	0.0
	$\sigma_{Y_2}^N$	$\sqrt{1.3}$ = 1.140	1.140	1.140	1.140	1.140	1.140
Step 5	$\frac{\partial g}{\partial Y_1}$	72.556	82.774	87.709	88.304	84.990	83.882
	$\frac{\partial g}{\partial Y_2}$	217.64	160.846	162.898	163.247	174.963	174.368
Step 6	α_{Y_1}	0.238	0.353	0.367	0.369	0.336	0.333
	α_{Y_2}	0.971	0.936	0.930	0.929	0.942	0.943
Step 7		Go to Step 3. Compute the new checking point using information from Step 6.					
Step 8	β				3.927		3.922
Step 9		Repeat Steps 3 through 8 until β converges.					

The final checking point is (23.732, 48.036).

8.3.2 Correlated Nonnormal Variables

EXAMPLE 8.3

Example 8.2 for correlated normal random variables can be considered again, except that the random variable F_y is a lognormal random variable with a mean of 38 ksi and a standard deviation of 3.8 ksi. The random variable Z is again normal, with mean and standard deviation of 54 in.^{3} and 2.7 in.^{3}, respectively. Assume further that F_y and Z are correlated with a correlation coefficient of 0.3.

In general, the limit state equation for correlated nonnormal variables is quite involved. Since FORM is an iterative procedure, at each iteration the checking point and the corresponding equivalent mean and standard deviation of nonnormal variables are expected to be different, indicating that the limit state equation needs to be redefined in each iteration. It is not necessary to give complete hand calculations to estimate the safety index here. However, all the necessary steps required to solve the problem are outlined as follows. The results are summarized in Table 8.3.

As an approximation, it is assumed that the covariance matrix [C], the correlation matrix [C′], and the corresponding eigenvalues, eigenvectors, and transformation

matrix [**T**] do not change with the distribution of the random variables. Thus, the eigenvalues of 0.7 and 1.3 and the corresponding [**T**] matrix obtained earlier when both F_y and Z are normal variables can be used in this case also. The steps of FORM Method 1 are illustrated next.

Iteration 1

Based on past experience with similar problems, it is expected that the safety index β will be closer to 5.0 than 3.0. Thus, β is assumed to be 5.0 in the first iteration. However, β can still be assumed to be 3.0, but it will take more iterations to reach convergence.

As before, the initial checking point values of F_y and Z can be assumed to be the same as their mean values: 38 and 54, respectively. Since F_y is lognormal with $\lambda_{F_y} = 3.632611$ and $\zeta_{F_y} = 0.0997513$, its equivalent mean and standard deviation at the checking point can be estimated as (See Alternative 2 in Example 7.6)

$$\sigma_{F_y}^N = 0.0997513 \times 38 = 3.791$$

and

$$\mu_{F_y}^N = 38(1 - \ln 38 + 3.632611) = 37.811.$$

Table 8.3 Steps in FORM Method 1 for Correlated Nonnormal Random Variables

Step 1				$g(\ ) = F_y Z - 1{,}140 = 0$			
Step 2	β	5.0			4.576		4.585
Step 3	f_y^*	38.	20.305	23.855	25.058	25.058	25.037
	z^*	54.	45.320	44.695	45.496	45.550	45.535
Step 4	$\mu_{F_y}^N$	37.811	32.930	34.843	35.367	35.367	35.359
	$\sigma_{F_y}^N$	3.791	2.025	2.380	2.500	2.500	2.498
	μ_Z^N	54.0	54.0	54.0	54.0	54.0	54.0
	σ_Z^N	2.7	2.7	2.7	2.7	2.7	2.7
Step 5	$\frac{\partial g}{\partial Y_1}$	72.231	26.139	29.688	32.605	32.689	32.622
	$\frac{\partial g}{\partial Y_2}$	217.317	103.654	120.762	128.269	128.360	128.212
Step 6	α_{Y_1}	0.2370	0.1820	0.1775	0.1834	0.1837	0.1835
	α_{Y_2}	0.9715	0.9833	0.9841	0.9830	0.9830	0.9830
Step 7		Go to Step 3. Compute the new checking point using information from Step 6.					
Step 8	β			4.576		4.585	4.586
Step 9		Repeat Steps 3 through 8 until β converges.					

The final checking point is (25.038, 45.531).

Then, using Equation 8.11, we can show that

$$\begin{Bmatrix} F_y \\ Z \end{Bmatrix} = \begin{bmatrix} 3.791 & 0 \\ 0 & 2.7 \end{bmatrix} \begin{bmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} + \begin{Bmatrix} 37.811 \\ 54 \end{Bmatrix}$$

or,

$$F_y = 2.681Y_1 + 2.681Y_2 + 37.811$$

$$Z = -1.909Y_1 + 1.909Y_2 + 54.$$

Thus, the limit state equation becomes

$$(2.681Y_1 + 2.681Y_2 + 37.811)(-1.909Y_1 + 1.909Y_2 + 54) - 1{,}140 = 0$$

or

$$g(\,) = -5.118\ Y_1^2 + 5.118\ Y_2^2 + 72.593\ Y_1 + 216.955\ Y_2 + 901.794 = 0.$$

Thus,

$$\frac{\partial g}{\partial Y_1} = -10.236Y_1 + 72.593$$

$$\frac{\partial g}{\partial Y_2} = 10.236Y_2 + 216.955.$$

The random variables F_y and Z in the reduced coordinates at the checking point become

$$F_y'^{*} = \frac{38 - 37.811}{3.791} = 0.050 \text{ and } Z'^{*} = \frac{54 - 54}{2.7} = 0$$

Since the transformation matrix $\mathbf{T}$ is orthogonal, $\mathbf{T}^{-1} = \mathbf{T}^t$. Using Equation 8.11, we can show that $\mathbf{Y} = \mathbf{T}^t\,\mathbf{X}'$. Thus, the coordinates of the checking point in the $\mathbf{Y}$ coordinates become

$$\begin{Bmatrix} y_1^* \\ y_2^* \end{Bmatrix} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} 0.050 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.0354 \\ 0.0354 \end{Bmatrix}.$$

Thus,

$$\left(\frac{\partial g}{\partial Y_1}\right)^* = -10.236 \times 0.0354 + 72.593 = 72.231$$

$$\left(\frac{\partial g}{\partial Y_2}\right)^* = 10.236 \times 0.0354 + 216.955 = 217.317.$$

The direction cosines of Y_1 and Y_2 can be shown to be

$$\alpha_{Y_1} = \frac{72.231 \times \sqrt{0.7}}{\sqrt{72.231^2 \times 0.7 + 217.317^2 \times 1.3}} = 0.2370$$

and

$$\alpha_{Y_2} = \frac{217.317 \times \sqrt{1.3}}{\sqrt{72.231^2 \times 0.7 + 217.317^2 \times 1.3}} = 0.9715.$$

Then,

$$y_1^* = -0.2370 \times \sqrt{0.7} \times 5.0 = -0.9914$$

$$y_2^* = -0.9715 \times \sqrt{1.3} \times 5.0 = -5.5384.$$

The new checking point in the original coordinate becomes

$$f_y^* = 2.681(-0.9914 - 5.5384) + 37.811 = 20.305$$

$$z^* = -1.909(-0.9914) + 1.909(-5.5384) + 54 = 45.320.$$

This completes the first iteration.

Second Iteration

With this new checking point, the equivalent normal mean and standard deviation of F_y are calculated again. Proceeding in a manner similar to the first iteration and using Equation 8.11, we can show the relationships between F_y and Z, and Y_1 and Y_2, to be

$$F_y = 1.432(Y_1 + Y_2) + 32.930$$

$$Z = 1.909(-Y_1 + Y_2) + 54.$$

The direction cosines of Y_1 and Y_2 are found to be 0.1820 and 0.9833, respectively. They do not converge to the values from the first iteration with a tolerance level of 0.005. However, they are found to be 0.1775 and 0.9841, respectively, after the third iteration satisfying the tolerance criterion, as shown in Table 8.3. With β as an unknown parameter, the checking point in the $\mathbf{Y}$ coordinates becomes

$$y_1^* = -0.1775 \times \sqrt{0.7}\beta = -0.1485\beta$$

and

$$y_2^* = -0.9841 \times \sqrt{1.3}\beta = -1.1220\beta.$$

The checking point in the original coordinates becomes

$$f_y^* = 1.683(-0.1485\ \beta - 1.1220\ \beta) + 34.843 = -2.1383\ \beta + 34.843$$

$$z^* = 1.909(0.1485\ \beta - 1.1220\ \beta) + 54 = -1.8584\ \beta + 54.$$

Substituting these in the limit state equation, we find β to be 4.576. This β value is not acceptable with a tolerance level of 0.005. With this new β value, a new checking point can again be defined. Proceeding as in the previous steps, we find the direction cosines of Y_1 and Y_2 to be 0.1834 and 0.9830, respectively. These values are not acceptable with a tolerance level of 0.005. However, in the next iteration, they converse to 0.1837 and 0.9830, respectively. The corresponding β value is found to be 4.585. Again, this is not acceptable with a tolerance level of 0.005.

When the new β value of 4.585, the direction cosines of Y_1 and Y_2 become 0.1835 and 0.9830, satisfying the tolerance criterion. The corresponding safety index is found to be 4.586, which satisfies the tolerance criterion. Thus, for this problem with correlated nonnormal variables, the safety index is found to be 4.586 and the corresponding checking point is (25.038, 45.531). The results are summarized in Table 8.3.

This example clearly indicates that hand calculations for this type of problem can be very cumbersome, but a computer program can be easily written to implement the algorithm.

8.4 PROBABILISTIC SENSITIVITY INDICES

Because all input random variables do not have equal influence on the statistics of the output, a measure called the *sensitivity index* can be used to quantify the influence of each basic random variable. The quantity $\nabla g(\mathbf{Y})$, which is the gradient vector of the

performance function in the space of standard normal variables, is used for this purpose. Let $\boldsymbol{\alpha}$ be a unit vector in the direction of this gradient vector. Then, because the design point can be expressed as $\mathbf{y}^* = -\beta\boldsymbol{\alpha}$, it is easily seen that

$$\alpha_i = -\frac{\partial \beta}{\partial y_i^*}. \tag{8.15}$$

Thus, the elements of the vector $\boldsymbol{\alpha}$ are directly related to the derivatives of β with respect to the standard normal variables. If these are related to the original variables and their statistical variation, a unit sensitivity vector can be derived as (Der Kiureghian and Ke, 1985)

$$\boldsymbol{\gamma} = \frac{\mathbf{S}\mathbf{B}^t\boldsymbol{\alpha}}{|\mathbf{S}\mathbf{B}^t\boldsymbol{\alpha}|} \tag{8.16}$$

where $\mathbf{S}$ is the diagonal matrix of standard deviations of the input variables (equivalent normal standard deviations for the nonnormal random variables) and $\mathbf{B}$ is also a diagonal matrix required to transform the original variables $\mathbf{X}$ to equivalent uncorrelated standard normal variables $\mathbf{Y}$. i.e., $\mathbf{Y} = \mathbf{A} + \mathbf{B}\,\mathbf{X}$. For the ith random variable, this transformation is $Y_i = (X_i - \mu_{X_i})/\sigma_{X_i}$. Thus, the matrix $\mathbf{B}$ contains the inverse of the standard deviations or the equivalent normal standard deviations. If the variables are statistically independent, then in Equation 8.16, the product of $\mathbf{S}\,\mathbf{B}^t$ will be a unit diagonal matrix. Thus, the sensitivity vector will be identical to the direction cosines vector of the random variables. However, if the variables are correlated, another transformation matrix $\mathbf{T}$, as in Equation 8.11, will come into the picture. Then, the sensitivity vector and the direction cosines vector will be different.

The elements of the vector $\boldsymbol{\gamma}$ may be referred to as sensitivity indices of individual variables. The sensitivity indices can be used to improve computational efficiency. Variables with very low sensitivity indices at the end of the first few iterations can be treated as deterministic at their mean values for subsequent iterations in the search for the minimum distance. This significantly reduces the amount of computation because, as a practical matter, only a few variables have a significant effect on the probability of failure. These sensitivity indices are also useful in reducing the size of problems with random fields, in which the random fields are discretized into sets of correlated random variables (Haldar and Mahadevan, 2000; Mahadevan and Haldar, 1991) and in reliability-based optimization (Mahadevan, 1992).

8.5 SYSTEM RELIABILITY EVALUATION

In the previous sections of this chapter, reliability was estimated for a single performance criterion or limit state using FORM or SORM. In general, any engineering system has to satisfy more than one performance criterion. Even for a simple beam, the performance criterion could be strength related (e.g., bending moment or shear) or serviceability related (e.g., deflection or vibration). Thus, the beam can fail in more than one performance mode. A structure such as a truss or a frame consists of multiple structural elements or components, and failure may occur in one or more components. The long-distance telephone communication system between the East and West Coasts consists of several networks. At any given time, one or more such networks may not be in operating condition. Ordinary service may or may not suffer disruption depend-

ing upon the availability of alternative systems. The water supply to a community may come from different sources through a network of piping. Again, the water supply from different sources and/or networks may be disrupted, but water may still be available to the community. The concept used to consider multiple failure modes and/or multiple component failures is known as *system reliability* evaluation. A complete reliability analysis includes both component-level and system-level estimates.

In general, system reliability evaluation is quite complicated and depends on many factors. Some of the important factors are (1) the contribution of the component failure events to the system's failure, (2) the redundancy in the system, (3) the postfailure behavior of a component and the rest of the system, (4) the statistical correlation between failure events, and (5) the progressive failure of components. Considering the beam example, we can estimate its reliability by calculating the probability of satisfying all the performance criteria. An engineering system usually consists of multiple components, and system failure may occur when one or more components fail. This is still an active area of research. Only the fundamental concepts essential for system reliability evaluation are introduced briefly in this section.

In the simplest case of a system that has multiple performance criteria, system failure may be defined as occurring when any of the criteria are violated. Such a system is referred to as a series system or weakest link system. In this case, system failure is defined by the union of the individual performance failures. Alternatively, a system might fail only when all the components fail, as in the case of a multiple cable system to carry a load. This is called a parallel or redundant system. In this case, system failure is defined by the intersection of the individual (component) failure events. Some systems may be defined through a combination of series and parallel connections of subsystems. For example, in a frame structure, each member might fail when it fails to satisfy one of several performance criteria (bending, shear, axial force, deflection, etc.), whereas the overall structure may fail when some (not necessarily all) of the members fail. In a system with redundancy, significant computational effort is needed to identify the ways in which the system fails or survives.

The application of the concept of system reliability in various engineering disciplines can be described, at best, as nonuniform. Most of the progress has been made in structural engineering, and in the following discussion, the application of the system reliability concept in structural engineering is emphasized.

Two basic approaches used for system reliability evaluation are the cut-set or failure mode approach (FMA) or performance mode approach (PMA), and the tie-set or stable configuration approach (SCA). In the FMA, all the possible ways a structure can fail are identified. A *fault tree* diagram, which decomposes the main failure event into unions and intersections of subevents or combinations of subevents, can be used for this purpose. Alternatively, an *event tree*, which systematically identifies the possible sequence of events, can also be used to identify the important failure sequences of the structure. Several sequences might lead to the same system failure mode. For example, if the failure of Components 1, 2, and 3 constitutes a system failure mode, the sequences could be 1–2–3, 1–3–2, 2–3–1, and so on. Some of these sequences may have a higher probability than others. A system is considered to fail if any of the failure sequences (or modes) occur. This leads to the definition of the system failure event as the union of several sequences (or failure modes defined by the sequences). Neglecting some of the

potential failure modes or sequences may result in an underestimation (unconservative) of the probability of system failure. The FMA is very effective for systems with ductile components (components that continue to carry loads after reaching their capacity), particularly when the dominant failure mechanisms of the system can be easily identified.

The SCA considers how a system or its damaged states can carry loads without failure. A system is not expected to fail if any of its states are stable under the applied loads. This leads to the intersection of survival (no failure) events. In the SCA, neglecting some of the stable configurations will result in the overestimation (conservative) of the probability of system failure. The SCA is effective for highly redundant systems with brittle components (components that fail to carry loads after reaching their capacity) or with ductile and brittle components.

Once the failure modes or stable configurations of a system are identified, system reliability evaluation involves evaluating the probability of union and intersection of events considering the statistical correlation between them. However, in many cases, the statistical correlation may be difficult to estimate. Also, as discussed in the following section, it is difficult to estimate the joint probabilities of more than two failure events. These difficulties result in an estimation of upper and lower bounds for the system reliability evaluation. These bounds are usually estimated by assuming that all the events are either perfectly correlated or statistically independent, as discussed in the following sections. The actual probability of failure of the system will be between these bounds. Several methods are available to reduce the width of these bounds. In the following sections, system reliability evaluation procedures are discussed for some simple but basic cases.

8.5.1 Series Systems or Weakest Link Systems

Consider the steel portal frame shown in Figure 8.4. All three members are assumed to have identical plastic moment capacity M_p. The frame is subjected to two loads, V and H, which are considered to be random variables and are assumed to follow the Gaussian distribution. Three types of system collapse mechanisms are considered: beam mechanism, sway mechanism, and combined mechanism. The limit states corresponding to each of these mechanisms are shown in Figure 8.5. The system failure domain is shown by the shaded region, which is the union of the failure domains for individual mechanisms. System failure probability is defined as the integral of the joint probability density function of the random variables V and H over the system failure domain.

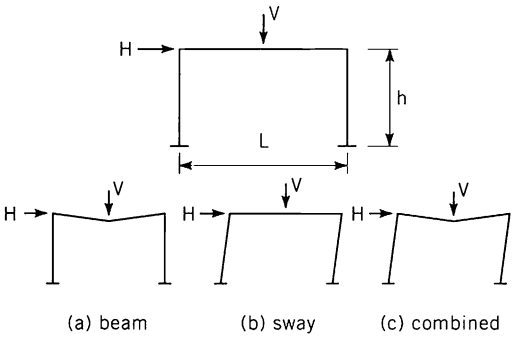


Figure 8.4 Portal Frame—Three Plastic Collapse Modes

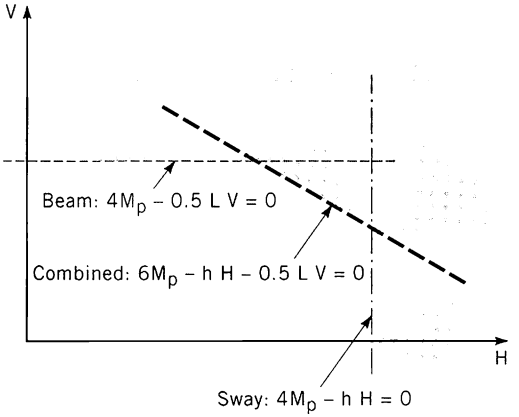


Figure 8.5 Portal Frame—Three Collapse Limit States

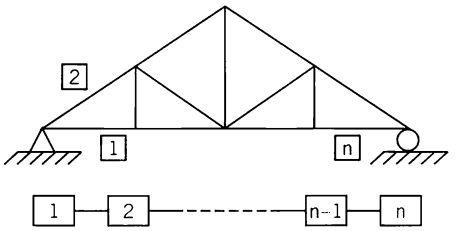


Figure 8.6 Statically Determinate Truss—Series or Weakest Link System

A statically determinate truss with n components is shown in Figure 8.6, where the failure of one or more components constitutes the failure of the truss. The multiple failure modes of the frame or the failure of one or more components of the truss represents the series or weakest link system.

With E_i representing the failure event of the ith component, the event of the failure of a series system can be defined by the union of all the component failure events. After the failure probabilities for the individual events are computed, determining the probability of the union of all the failure modes is a difficult problem. Consider the probability of the union of three failure events E_1, E_2, and E_3, defined as

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1E_2) - P(E_2E_3) - P(E_3E_1) + P(E_1E_2E_3) \tag{8.17}$$

where $P(E_1 E_2)$ refers to the joint probability of E_1 and E_2. Other joint probability terms are similarly obvious from the equation. Because it is difficult to determine the joint probabilities of more than two failure events except by using Monte Carlo simulation or numerical integration, several approximate bounds have been proposed for the system failure probability. The simplest among these are the first-order bounds suggested by Ang and Amin (1967) and Cornell (1967). With p_{fs} denoting the probability of failure of the system, these first-order bounds are

$$\max[P(E_i)] \le p_{fs} \le \min\left[\sum_{i=1}^{n} P(E_i),\ 1\right] \tag{8.18}$$

where $P(E_i)$ is the probability of failure of the ith failure event, and n is the number of failure events. In Equation 8.18, the lower bound represents the system failure

probability if all the events are perfectly dependent, and the upper bound represents the system failure probability if all the events are mutually exclusive. If the events are statistically independent, the upper bound becomes

$$p_{fs} = \leq 1 - \prod_{i=1}^{n} [1 - P(E_i)]. \tag{8.19}$$

The foregoing first-order bounds could be quite wide; for accurate estimation, second-order bounds are used. These include the second-order terms in Equation 8.17, that is, the joint probabilities of two events. The following second-order bounds are widely used (Ditlevsen 1979b):

$$P(E_1) + \sum_{i=2}^{n} \max\left\{\left[P(E_i) - \sum_{j=1}^{i-1} P(E_i E_j)\right], 0\right\} \leq p_{fs} \leq \min\left\{\left[\sum_{i=1}^{n} P(E_i) - \sum_{i=2}^{n} \max_{j<i} P(E_i E_j)\right], 1\right\}. \tag{8.20}$$

Experience shows that ranking the individual events in the order of decreasing probability will give the narrowest bounds in this formula. In that case, $P(E_1)$ is the probability of the most probable event.

EXAMPLE 8.4

Consider the five-member statically determinate truss shown in Figure 2.5. For this illustration, assume that the probability of failure of each of the five members is 0.001, 0.002, 0.003, 0.004, and 0.005, respectively. Calculate the first-order bounds for the probability of failure of the truss.

SOLUTION

Using Equation 8.18, we can show the first-order bounds of the probability of failure of the truss to be

$$0.005 \leq p_{fs} \leq 0.015.$$

If the failures of the members are assumed to be statistically independent, then using Equation 8.19, we find the first-order bounds to be

$$0.005 \leq p_{fs} \leq 1 - (1 - 0.001)(1 - 0.002)(1 - 0.003)(1 - 0.004)(1 - 0.005)$$

or

$$0.005 \leq p_{fs} \leq 0.0149.$$

EXAMPLE 8.5

A beam can fail by exceeding the bending moment capacity, by exceeding the shear force capacity, or by excessive deflection. The corresponding probabilities of failure are calculated to be 10^{-3}, 10^{-5}, and 10^{-6}, respectively. Calculate the probability of failure of the beam.

SOLUTION

Equation 8.18 is used to find the first-order bounds of the probability of failure of the beam:

$$10^{-3} \leq p_{fs} \leq 1.011 \times 10^{-3}$$

The two bounds are very narrow in this example, indicating that if the probabilities of failures are very different from each other, the maximum probability of failure will control the design.

8.5.2 Parallel Systems

A parallel (or redundant or fail-safe) system is one in which system failure requires the occurrence of all the individual failure events. In this case, the probability of system failure is the probability of the joint occurrence of all the individual failure events. Suppose n cables are carrying a load of P as shown in Figure 8.7. In this case, all cables must fail to have a system failure. Thus, unlike the series system case where the unions of the failure events are considered, in a parallel system the intersections of the failure events are considered. The bounding methods can still be used; however, the bounds for the intersection of failure events will not, in general, be as close as the bounds for the union of failure events (Bennett and Ang, 1983). The first-order bounds for the parallel system are

$$\max\left\{1 - \sum_{i=1}^{n}[1 - P(E_i)],\ 0\right\} \leq p_{fs} \leq \min[P(E_i)]. \tag{8.21}$$

In most cases, the lower bound will be close to zero. The upper bound is exact if the failure events are perfectly correlated. For statistically independent events, the lower bound becomes

$$\prod_{i=1}^{n} P(E_i) \leq p_{fs}. \tag{8.22}$$

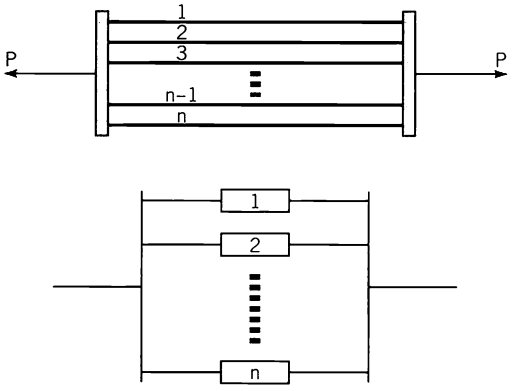


Figure 8.7 Parallel System

The second-order bounds for the parallel system (Quek and Ang, 1986) are

$$\max\left\{\prod_{i=1}^{n} P(E_i),\ \left[\sum_{i=1}^{n} P(E_i) - \sum_{i=2}^{n} \min_{j<i} P(E_i \cup E_j)\right]\right\} \le p_{fs} \le$$
$$P(E_1) - \sum_{i=2}^{n} \max\left\{\left[2 - i - P(E_i) + \sum_{j=1}^{i-1} P(E_i \cup E_j)\right],\ 0\right\}. \tag{8.23}$$

Usually, the upper bound of system failure probability is of interest for risk assessment, design decisions, and resource allocation for practical problems. The upper bound in Equation 8.23 is one of several versions of the upper bound proposed by Ditlevsen (1979b) as

$$P_{fs} \le 1 - \left\{P(\overline{E}_1) + \sum_{i=2}^{n} \max\left[0,\ P(\overline{E}_i) - \sum_{j=1}^{i-1} P(\overline{E}_i \cap \overline{E}_j)\right]\right\}. \tag{8.24}$$

This bound is order-dependent. For best results, the individual events should be numbered in increasing order in terms of their failure probabilities, that is,

$$P(E_1) < P(E_2) < \ldots\ldots\ldots < P(E_n).$$

Other equivalent forms of Ditlevsen's bounds are given by Thoft-Christensen and Murotsu (1986) and Karamchandani (1987). Xiao and Mahadevan (1994c) showed the equivalence of these various forms and showed that for problems where the individual failure probabilities of the events are less than 0.25 (which is usually true for practical problems), Ditlevsen's upper bound reduces to a very simple formula:

$$P_{fs} \le P(E_1 \cap E_2). \tag{8.25}$$

That is, the joint probability of the two least probable failure events provides a good second-order upper bound for practical problems, requiring very little computation. In fact, one can consider this simple formula as the first of three options increasing in accuracy and computational effort. The second option is

$$P_{fs} \le \min[P(E_1 \cap E_j)], \quad j = 2, \ldots\ldots\ldots, n. \tag{8.26}$$

That is, compute the joint probability of the least probable event E_1 with all other events and take the minimum of these joint probabilities as the upper bound. The third option was given by Murotsu et al. (1981) as

$$P_{fs} \le \min[P(E_i \cap E_j)], \quad i \ne j, \quad i, j = 1, 2, \ldots\ldots\ldots, n. \tag{8.27}$$

This is also computationally the most intensive, since it requires the computation of joint failure probabilities of all two-event intersections and then using the minimum of these as the upper bound.

EXAMPLE 8.6

For the safe operation of nuclear power plants, the power supply must be available at all times. To ensure this, several levels of redundancy are incorporated in the power supply system. In case a plant fails to generate any power, it is connected to another plant. In case this offsite source of power is not available when needed, multiple diesel generators are placed onsite, which are designed to start operating when other sources

of power are unavailable. If the diesel generators fail to start, batteries are available to supply the necessary power to at least shut down the plant. In this case, the offsite and onsite power sources can be considered to constitute a parallel system. The plant will face catastrophe only if all four sources of power are unavailable at a given time.

Suppose the probabilities that power will not be available from the plant itself, the offsite plant, the diesel generators, and the batteries are 0.01, 0.05, 0.002, and 0.005, respectively. Assuming all four sources of power supply are statistically independent of each other, and using Equations 8.21 and 8.22, we find the first-order bounds to be

$$(0.01)(0.05)(0.002)(0.005) \le p_{fs} \le 0.002$$

or

$$5 \times 10^{-10} \le p_{fs} \le 0.002.$$

8.5.3 Nonlinear System Reliability

The individual failure probabilities in Equations 8.18 to 8.27 are estimated using a first-order or second-order approximation to the limit state in a space that has been obtained through an approximate equivalent normal transformation of the basic random variables. Therefore, the "bounds" are not the true bounds of the system failure probability. They become more approximate as one begins to compute the joint probabilities of two events, as described next.

To use second-order bounds, the joint probability of two events $P(E_i\, E_j)$ needs to be estimated. A first-order approximation of the joint probability, constructed by Ditlevsen (1979b), is illustrated geometrically in Figure 8.8 for two linear limit states. The individual probabilities of failure events in the first-order analysis are determined as

$$P(E_i) = \Phi(-\beta_i). \tag{8.28}$$

In the equivalent standard normal space, each limit state may be represented as

$$g_i(\,) = \beta_i + \sum_{r=1}^{m} \alpha_r Y_r \tag{8.29}$$

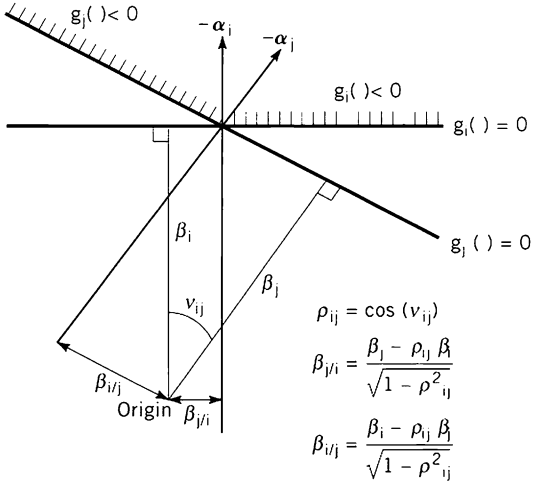


Figure 8.8 Joint Probability of Two Failure Modes

where m is the number of random variables. The angle between two limit states provides information about the correlation between the two failure modes. Thus, the correlation coefficient is obtained as

$$\rho_{ij} = \sum_{r=1}^{m} \alpha_{ir}\alpha_{jr} = \cos \nu_{ij}. \tag{8.30}$$

Once β_i, β_j, and ρ_{ij} are known, the joint probability of failure can be calculated as (Madsen et al., 1986)

$$P(E_i \cap E_j) = \Phi_2(-\beta_i, -\beta_j, \rho_{ij}) \tag{8.31}$$

where Φ_2 refers to the joint cumulative distribution of two standard normal variables. This is computed as

$$\Phi_2(-\beta_i, -\beta_j, \rho_{ij}) = \Phi(-\beta_i)\Phi(-\beta_j) + \int_0^{\rho} \phi_2(-\beta_i, -\beta_j, t)dt \tag{8.32}$$

where

$$\phi_2(-\beta_i, -\beta_j, t) = \frac{1}{2\pi\sqrt{1-t^2}} \exp\left(-\frac{1}{2}\frac{\beta_i^2 - 2t\beta_i\beta_j + \beta_j^2}{1-t^2}\right) \tag{8.33}$$

is the joint probability density function of two standard normal variables. A similar general formula to compute the joint probability of more than two events was provided by Hohenbichler and Rackwitz (1983), and its accuracy was further improved by Gollwitzer and Rackwitz (1988) using asymptotic analysis.

This method for joint probability estimation could be inaccurate for nonlinear limit states, as shown in Figure 8.9. In such cases, one needs to find the exact intersection between the two limit states and then estimate the joint probability by constructing linear approximations of the limit states at this intersection. In a multidimensional random variable space, the intersection is a hypersurface. Therefore, the minimum distance point on this intersection hypersurface is found and linear approximations of the two limit states are constructed at this point, which may be referred to as the joint most probable point (joint MPP).

The joint MPP can be estimated using a nonlinear programming algorithm in which the objective is to minimize the distance subject to two limit state constraints.

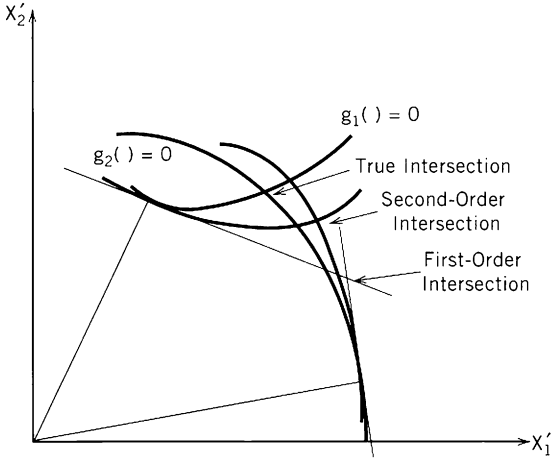


Figure 8.9 Two Nonlinear Limit States

Alternatively, the Rackwitz–Fiessler algorithm can be extended to the case of two limit states to achieve this objective (Mahadevan and Cruse, 1992).

Once the individual mode failure probabilities and the two-mode joint failure probabilities are computed, approximate second-order bounds for the system failure probability can be computed as discussed previously.

8.6 IMPLICIT PERFORMANCE FUNCTIONS

In the discussions in Chapters 7 and 8 on reliability methods, we mention that risk or reliability is calculated for a given performance criterion or a performance function. However, for most realistic structures or systems, the response has to be computed through a numerical procedure such as finite element analysis. This brings another level of complexity to reliability analysis because the performance function $g(\mathbf{X})$ is not available as an explicit, closed-form function of the input variables. In such cases, the derivatives of $g(\mathbf{X})$ with respect to the random variables $\mathbf{X}$, required in searching for the minimum distance point on the limit state, are not readily available.

Several computational approaches could be pursued for the reliability analysis of structures with implicit performance functions. These can be broadly divided into three categories, based on their essential philosophy: Monte Carlo simulation (including efficient sampling methods and variance reduction techniques), the response surface approach, and sensitivity-based analysis.

Monte Carlo simulation is discussed in Chapter 9. The examples in Chapter 9 feature closed-form explicit performance functions. However, simulation can also be used for problems with implicit performance functions, as long as an algorithm is available to compute the structural response. However, if the deterministic structural analysis is time-consuming, as with structures requiring finite element analysis, Monte Carlo simulation may become impractical.

With the response surface approach, a polynomial is constructed to approximate $g(\mathbf{X})$ through a few selected simulations in the neighborhood of the most likely failure point, and then a regression analysis of these results is performed or a set of linear equations is solved. Once the approximate closed-form representation of the limit state function is available, the FORM or SORM reliability methods discussed earlier can be used to estimate the failure probability (Haldar and Huh, 1999).

In the sensitivity-based approach, the sensitivity of the structural response to the input variables is computed and used in the FORM/SORM discussed earlier. The value of the performance function is calculated from deterministic analysis and the gradient is computed using sensitivity analysis. In the case of an explicit performance function, the gradient is computed simply by analytical or numerical differentiation of the performance function with respect to each random variable, as discussed in FORM Method 1. In the case of problems with an implicit performance function, FORM Method 2 can be used. Several approximation methods are also available to compute the gradient of the performance function.

The sensitivity-based reliability analysis approach is more elegant and, in general, more efficient than the simulation or response surface methods. This approach is also efficient in that it considers some of the random variables as deterministic if their importance on the reliability estimation is marginal.

The response sensitivities can be computed in three different ways: (1) through a finite difference approach, by perturbing each variable and computing the corresponding change in response through multiple deterministic analyses; (2) through classical perturbation methods that apply the chain rule of differentiation to finite element analysis; and (3) through iterative perturbation analysis techniques. All these methods have their own domains of usefulness.

Reliability estimation when the performance function is implicit, in general, is quite involved. The subject is discussed in detail by the authors in another book (*Reliability Assessment Using Stochastic Finite Element Analysis*, by Haldar and Mahadevan, John Wiley & Sons, New York, NY, 2000) in the context of the stochastic finite element method (SFEM) or probabilistic finite element method (PFEM). Interested readers are referred to this book.

8.7 CONCLUDING REMARKS

Some advanced topics in reliability estimation are presented in this chapter, and the concept of the second-order reliability method is introduced. Correlated random variables are often present in reliability analysis. The additional steps necessary for correlated variables are described in this chapter. Not all the variables in a problem may need to be considered random. To evaluate their relative importance in the overall reliability evaluation, the concept of probablistic sensitivity indices is introduced. Variables whose sensitivity indices are relatively low at the end of the first few iterations can be treated as deterministic, essentially reducing the size of the problem.

In the previous discussions in this book, reliability was estimated for a single performance criterion of a single component or element. The concept of system reliability for multiple failure modes and/or multiple component failures is introduced in this chapter. A complete reliability analysis includes both component-level and system-level estimates.

Reliability evaluation procedures when performance functions are implicit are briefly introduced. However, the problem is addressed in more detail elsewhere (Haldar and Mahadevan, 2000).

The information presented in the first eight chapters of this book provides readers with the necessary background to calculate the probability of failure or the reliability of simple systems. However, to verify the reliability results of analytical methods, simulation is essential. The concept of simulation is introduced in the next chapter.

8.8 PROBLEMS

8.1 Consider Problem 7.6. Use SORM to calculate the probability of failure.

8.2 Consider Problem 7.8. Use SORM to calculate the probability of failure.

8.3 Consider Problem 7.11. Use SORM to calculate the probability of failure.

8.4 Consider Problem 7.5. If w and P are correlated normal variables with $\rho_{wP} = 0.7$, with all other information remaining the same, use FORM to calculate the safety index.

8.5 In Problem 8.4, if w is a normal radom variable and P is a lognormal random variable with the same means and standard deviations, and the correlation coefficient between them is still 0.7, use FORM to calculate the safety index.

8.6 In Problem 8.4, if w and P are correlated lognormal variables with $\rho_{wP} = 0.7$, with all other information remaining the same, calculate the safety index using FORM.

8.7 The quality of water in a stream can be studied by considering the turbidity (T), concentration of organic materials (O), and concentration of other hazardous materials (H). The probabilities of high T, O, and H are 10^{-4}, 10^{-4}, and 10^{-5}, respectively.

(a) Calculate the first-order bounds of the probability of poor water quality in the stream.

(b) If T, O, and H are assumed to be statistically independent, how will the bounds obtained in Part (a) change?

8.8 To go to an airport, a passenger can use a personal car (C), the subway (S), or a taxicab (T). On a particular winter morning, these modes of transportation may be unavailable with probabilities of 0.001, 0.0001, and 0.0001, respectively.

(a) Calculate the first-order bounds that the passenger will not be able to go to the airport.

(b) If C, S, and T are assumed to be statistically independent, how will the bounds in Part (a) change?

Chapter 9

Simulation Techniques

9.1 INTRODUCTORY COMMENTS

Several methods with various degrees of complexity that can be used to estimate the reliability or safety index or the probability of failure are discussed in Chapters 7 and 8. Many of these methods are applicable when the limit state equations are explicit functions of the random variables involved in a problem. In FORM Method 2 presented in Chapter 7, we also introduce the concept of reliability evaluation when the limit state equations are implicit functions of the random variables involved in a problem. The reliability estimation procedures when the limit states are implicit are discussed in more detail elsewhere (Haldar and Mahadevan, 2000). In any case, estimating the probability of failure using these techniques requires a background in probability and statistics, as discussed in the previous chapters of this book. But with a simple simulation technique, it is possible to calculate the probability of failure for both the explicit or implicit limit state functions without knowing these analytical techniques and with only a little background in probability and statistics. The availability of personal computers and software makes the process very simple. In fact, to evaluate the accuracy of these sophisticated techniques or to verify a new technique, simulation is routinely used to independently evaluate the underlying probability of failure.

In the simplest form of the basic simulation, each random variable in a problem is sampled several times to represent its real distribution according to its probabilistic characteristics. Considering each realization of all the random variables in the problem produces a set of numbers that indicates one realization of the problem itself. Solving the problem deterministically for each realization is known as a *simulation cycle, trial,* or *run*. Using many simulation cycles gives the overall probabilistic characteristics of the problem, particularly when the number of cycles N tends to infinity. The simulation technique using a computer is an inexpensive way (compared to laboratory testing) to study the uncertainty in the problem.

9.2 MONTE CARLO SIMULATION TECHNIQUE

The method commonly used for this purpose is called the *Monte Carlo simulation technique*. The name itself has no significance, except that it was used first by von Neumann during World War II as a code word for nuclear weapons work at the Los Alamos National Laboratory in New Mexico. Most commonly the name Monte Carlo is associated with a place where gamblers take risks. This technique has evolved as a very powerful tool for engineers with only a basic working knowledge of probability and statistics for evaluating the risk or reliability of complicated engineering systems.

The Monte Carlo simulation technique has six essential elements: (1) defining the problem in terms of all the random variables; (2) quantifying the probabilistic characteristics of all the random variables in terms of their PDFs or PMFs and the corresponding parameters; (3) generating the values of these random variables; (4) evaluating the problem deterministically for each set of realizations of all the random variables, that is, numerical experimentation; (5) extracting probabilistic information from N such realizations; and (6) determining the accuracy and efficiency of the simulation. All these elements are discussed in the following section. Initially, all the random variables are considered to be uncorrelated. The use of the Monte Carlo simulation technique for correlated random variables is discussed in Section 9.4.

9.2.1 Formulation of the Problem

Consider a simply supported beam, shown in Figure 9.1, subjected to a uniformly distributed load W and a concentrated load P at the midspan. Assume that both W and P are random variables and thus the design bending moment M at the midspan is also a random variable. The task is to evaluate the probabilistic characteristics of the design bending moment using the Monte Carlo simulation technique. If the span of the beam is 30 feet, the expression for the design bending moment can be written as

$$M = WL^2/8 + PL/4$$

or

$$M = 112.5W + 7.5P. \tag{9.1}$$

9.2.2 Quantifying the Probabilistic Characteristics of Random Variables

Detailed discussions are presented in Chapters 3 and 5 on how to determine the underlying distribution of a random variable in terms of its CDF and/or PDF or PMF and the corresponding parameters to define it uniquely. No additional discussion is necessary at this stage.

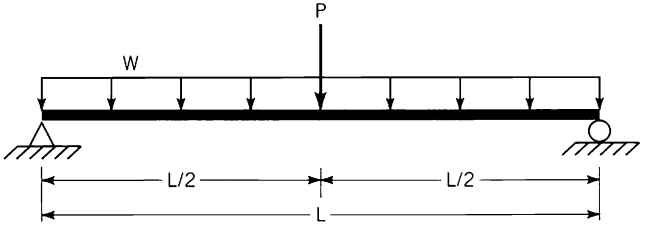


Figure 9.1 Simply Supported Beam

9.2.3 Generation of Random Numbers

The random variable to be generated could be continuous or discrete. Although the same concept underlies the generation of random numbers for continuous and discrete random variables, they need to be discussed separately.

9.2.3.1 Generation of Random Numbers for Continuous Random Variables

Assume W is a normal random variable with $\mu_W = 2$ kip/ft and $\sigma_W = 0.2$ kip/ft, and P is a uniformly distributed random variable between 10 and 20 kip. Further assume that they are statistically independent random variables. Of course, they could have any distribution, and one of them could be a known deterministic constant. If both of them are constants, then the bending moment is a constant and probabilistic study is not necessary. The task now is to generate N random numbers for W according to its probabilistic characteristics (i.e., in this case, a normal distribution with specified mean and standard deviation) and another N random numbers for P, which is uniformly distributed.

The generation of random numbers according to a specific distribution is the heart of Monte Carlo simulation. In general, all modern computers have the capability to generate uniformly distributed random numbers between 0 and 1. Sometimes the random number generators use bits and binary digits, and in most cases they are linear congruential generators. Corresponding to an arbitrary *seed value,* the generators will produce the required number of uniform random numbers between 0 and 1. By changing the seed value, different sets of random numbers can be generated. Depending upon the size of the computer, the random numbers may be repeated. However, this repetition will usually start only after generating a very large quantity of random numbers, such as 10^9. Random numbers generated this way are called *pseudo random numbers.* From a practical point of view, random numbers are rarely needed in this quantity; thus the repetition of random numbers is of academic interest only. One hundred random numbers for a uniform distribution between 0 and 1 are given in Table 9.1. These random numbers will be used in the subsequent discussion.

The next task is to transform the uniform random numbers u_i between 0 and 1, either generated by a computer or obtained from a table, to random numbers with the appropriate characteristics. The process is shown graphically in Figure 9.2. This is commonly known as the *inverse transformation technique* or *inverse CDF method.* In this method, the CDF of the random variable is equated to the generated random number u_i, that is, $F_X(x_i) = u_i$, and the equation is solved for x_i as

$$x_i = F_X^{-1}(u_i). \tag{9.2}$$

A simple example to describe the technique is the transformation of a uniform random number U between 0 and 1, such as $u_1 = 0.86061$ (the first number in Table 9.1), to another uniform random number x_1 between two limits a and b. The CDF of U is u_i. Since X is uniform, its CDF will be $F_X(x) = (x - a)/(b - a)$. The transformation to obtain the corresponding x_i value can be accomplished by equating the two CDFs as

$$u_i = \frac{x_i - a}{b - a}$$

or

$$x_i = a + (b - a)u_i. \tag{9.3}$$

Table 9.1 Uniform Random Numbers Between 0 and 1

0.86061	0.15017	0.42172	0.48932
0.92546	0.74098	0.95349	0.54707
0.41806	0.58515	0.16119	0.64271
0.28964	0.70074	0.58394	0.66930
0.14225	0.09666	0.95626	0.27681
0.44961	0.97948	0.20661	0.90451
0.24653	0.65400	0.24566	0.79163
0.21687	0.67980	0.94934	0.42397
0.56503	0.46872	0.16118	0.68086
0.40015	0.12846	0.01988	0.82174
0.83771	0.12237	0.27493	0.94600
0.73006	0.17468	0.03348	0.26457
0.56341	0.21305	0.38943	0.31697
0.82178	0.82744	0.36283	0.05336
0.32715	0.20220	0.41536	0.82238
0.68853	0.98479	0.30607	0.97673
0.74358	0.53164	0.14563	0.72927
0.24672	0.58442	0.44542	0.68251
0.90324	0.11799	0.53053	0.23987
0.79263	0.29124	0.58757	0.02894
0.44281	0.73958	0.17326	0.87885
0.70826	0.51527	0.10593	0.80716
0.22664	0.63765	0.72448	0.14197
0.62557	0.52224	0.44245	0.74708
0.48342	0.46079	0.37091	0.80193

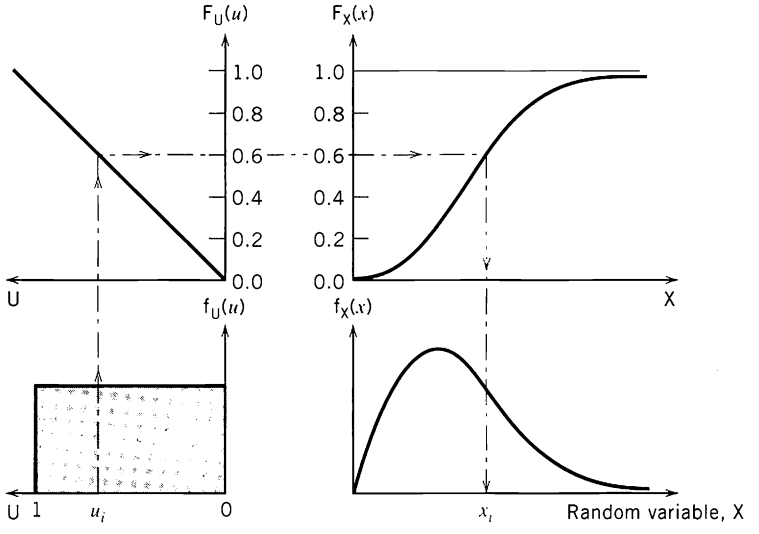


Figure 9.2 Mapping for Simulation

When $a = 0$ and $b = 1$, $x_i = u_i$, which is obvious. If X is uniform between 10 and 20, the corresponding first random number is

$$x_1 = 10 + (20 - 10) \times 0.86061 = 18.6061.$$

If X is normally distributed, that is, $N(\mu_X, \sigma_X)$, then $S = (X - \mu_X)/\sigma_X$ is a standard normal variate, that is, $N(0, 1)$, as discussed in Section 4.2.1. It can be shown that

$$u_i = F_X(x_i) = \Phi(s_i) = \Phi\left(\frac{x_i - \mu_X}{\sigma_X}\right) \tag{9.4}$$

or

$$s_i = \frac{x_i - \mu_X}{\sigma_X}.$$

Thus,

$$x_i = \mu_X + \sigma_X s_i = \mu_X + \sigma_X \Phi^{-1}(u_i). \tag{9.5}$$

Equation 9.5 suggests that in this case, the u_i values first need to be transformed to s_i, that is, $s_i = \Phi^{-1}(u_i)$, and Φ^{-1} is the inverse of the CDF of a standard normal variable.

Table 9.2 Standard Normal Random Numbers Corresponding to the Uniform Numbers in Table 9.1

1.08306	–1.03571	–0.19750	–0.02677
1.44279	0.64637	1.67968	0.11826
–0.20686	0.21509	–0.98958	0.36571
–0.55444	0.52653	0.21198	0.43798
–1.07027	–1.30081	1.70884	–0.59234
–0.12665	2.04313	–0.81824	1.30769
–0.68545	0.39614	–0.68821	0.81209
–0.78281	0.46714	1.63849	–0.19175
0.16373	–0.07849	–0.98962	0.47011
–0.25296	–1.13370	–2.05621	0.92202
0.98509	–1.16322	–0.59797	1.60725
0.61299	–0.93583	–1.83193	–0.62932
0.15962	–0.79588	–0.28081	–0.47619
0.92217	0.94410	–0.35090	–1.61313
–0.44780	–0.83379	–0.21378	0.92447
0.49169	2.16458	–0.50702	1.99047
0.65442	0.07939	–1.05536	0.61061
–0.68485	0.21321	–0.13724	0.47473
1.30024	–1.18509	0.07660	–0.70672
0.81558	–0.54977	0.22130	–1.89656
–0.14385	0.64205	–0.94136	1.16926
0.54831	0.03829	–1.24847	0.86748
–0.74996	0.35218	0.59620	–1.07151
0.32014	0.05578	–0.14476	0.66533
–0.04157	–0.09844	–0.32944	0.84854

Table 9.2 shows the set of 100 standard normal random numbers corresponding to the uniform random numbers between 0 and 1 given in Table 9.1. The x_i values can be calculated from the information on the s_i values. For $u_1 = 0.86061$, $s_1 = \Phi^{-1}(0.86061) = 1.08306$; with the information on μ_X and σ_X, the corresponding x_1 can be calculated. For the uniform load under consideration, the first random number according to the normal distribution is

$$x_1 = 2 + 0.2 \times 1.08306 = 2.21661.$$

If the random variable X is lognormally distributed with parameters λ_X and ζ_X, then the ith random number x_i according to the lognormal distribution can be generated as

$$u_i = \Phi\left(\frac{\ln x_i - \lambda_X}{\zeta_X}\right)$$

or

$$\ln x_i = \lambda_X + \zeta_X \Phi^{-1}(u_i)$$

or

$$x_i = \exp[\lambda_X + \zeta_X \Phi^{-1}(u_i)]. \tag{9.6}$$

A computer program can be written to generate random numbers according to any distribution. In fact, many available computer programs can generate random numbers for commonly used distributions. If the computer cannot generate a specific distribution, Equation 9.2 can be used to obtain it.

EXAMPLE 9.1

Generate random numbers for the Type II extreme value distribution discussed in Section 4.5.5, whose CDF is given by Equation 4.45:

$$F_X(x) = \exp\left[-\left(\frac{u}{x}\right)^k\right]$$

where u and k are the parameters of the distribution.

SOLUTION

Assuming that an available computer program cannot generate random numbers for the Type II extreme value distribution, they can be easily generated using Equation 9.2, as shown next. Denoting u_i as a uniform random number between 0 and 1, then

$$u_i = F_X(x_i) = \exp\left[-\left(\frac{u}{x_i}\right)^k\right]$$

or

$$x_i = \frac{u}{\left(\ln \frac{1}{u_i}\right)^{1/k}}. \tag{9.7}$$

Thus, for any u_i, the corresponding x_i according to the Type II extreme value distribution can be calculated using Equation 9.7.

9.2.3.2 Generation of Random Numbers for Discrete Random Variables

If X is a discrete random variable, its CDF $F_x(x_i)$ needs to be calculated by taking the summation of the individual PMFs, as in Equation 3.12. The inverse transformation technique can be used to generate discrete random numbers. It is necessary to equate u_i to the corresponding $F_X(x)$ value. Thus, it is necessary to evaluate the CDF for all possible values of X. Then, a numerical search procedure is needed to obtain the discrete random number, by satisfying the condition

$$F_X(x_{j-1}) < u_i \leq F_X(x_j) \tag{9.8}$$

EXAMPLE 9.2

The CDF of a binomial distribution is shown in Figure 3.4. Suppose $u_i = 0.86061$, the first uniform number between 0 and 1 in Table 9.1. Figure 3.4 indicates that

$$F_X(2) < 0.86061 < F_X(3).$$

In this case $x_i = 3$.

Generalizing the procedure, we can state that if X is a discrete random variable with CDF of $F_X(x_i)$, then

$$x_i \text{ is such that } i \text{ is the smallest integer with } u \leq F_X(x_i). \tag{9.9}$$

Generating discrete random numbers can be cumbersome in many cases. Several other procedures are available to generate discrete random numbers in a computer environment (Abramowitz and Stegum, 1964). They should be used whenever possible.

The discussion clearly indicates that any quantity of random numbers, discrete or continuous, according to specific CDFs can be generated from the information on uniform random numbers between 0 and 1.

9.2.4 Numerical Experimentation

N random numbers for each of the random variables in the problem will give N sets of random numbers, each set representing a realization of the problem. Thus, solving the problem N times deterministically will give N sample points, essentially generating information on the randomness in the output or response of the system to each set of input variables. The N generated sample points for the output or response can then be used to calculate all the required sample statistics, the histogram, the frequency diagram, the PDF or PMF and the corresponding CDF, and the probability of failure considering various performance criteria. The accuracy of the evaluation will increase as the number of simulations N increases. This is illustrated next.

EXAMPLE 9.3

The probabilistic characteristics of the bending moment for the beam shown in Figure 9.1 and represented by Equation 9.1 can now be generated using the Monte Carlo simulation technique. For the sake of brevity, only 10 simulation cycles are considered here. Again, W is N(2 kip/ft, 0.2 kip/ft) and P is uniform between 10 and 20 kip. Since

Table 9.3 Monte Carlo Simulations

$W \sim N(2, 0.2)$			P Uniform between 10 and 20		$m_i = 112.5\, w_i + 7.5\, p_i$
u_i	s_i	w_i	u_i	p_i	m_i
0.86061	1.08306	2.21661	0.83771	18.3771	387.1972
0.92546	1.44279	2.28856	0.73006	17.3006	387.2173
0.41806	–0.20686	1.95863	0.56341	15.6341	337.6014
0.28964	–0.55444	1.88911	0.82178	18.2178	349.1587
0.14225	–1.07027	1.78595	0.32715	13.2715	300.4553
0.44961	–0.12665	1.97467	0.68853	16.8853	348.7902
0.24653	–0.68545	1.86291	0.74358	17.4358	340.3459
0.21687	–0.78281	1.84344	0.24672	12.4672	300.8909
0.56503	0.16373	2.03275	0.90324	19.0324	371.4270
0.40015	–0.25296	1.94941	0.79263	17.9263	353.7557

Mean of moment = 347.68 kip-ft; standard deviation of moment = 30.38 kip-ft.

P is uniformly distributed, its mean value and COV can be calculated from Table 4.2. In this case, the mean value is $\mu_P = (10 + 20)/2 = 15$ kip and the COV is $\delta_P = 2/\sqrt{12} \times (20 - 10)/(20 + 10) = 0.1925$, and the corresponding standard deviation is $\sigma_P = 0.1925 \times 15 = 2.89$ kip.

Suppose 10 uniform random numbers between 0 and 1 (the first 10 numbers in Table 9.1) are generated for W and another 10 (the next 10 numbers, i.e. numbers 11 through 20 in Table 9.1) are generated for P. The steps involved in generating a set of 10 random numbers for w_i and p_i according to their statistical characteristics and the corresponding m_i are summarized in Table 9.3.

Using the 10 sample points for the bending moment thus generated, we can calculate its mean and standard deviation to be 347.68 k-ft and 30.38 k-ft, respectively. The statistical distribution of the bending moment can also be obtained by generating enough data to draw a histogram or by using other statistical techniques discussed in Chapter 5. The optimal numbers of simulation cycles required are discussed in Section 9.2.6.

If the mean value first-order second moment (MVFOSM) method discussed in Section 7.6.1 is used, the mean value and standard deviation of the bending moment are estimated as

$$\mu_M \approx 112.5 \times 2 + 7.5 \times 15 = 337.5 \text{ kip-ft}$$

and

$$\sigma_M \approx \sqrt{112.5^2 \times 0.2^2 + 7.5^2 \times 2.89^2} = 31.24 \text{ kip-ft}.$$

The differences between the parameters estimated by the two methods are expected to become narrower as the number of simulation cycles increases. The first-order approximation does not use the information on distributions of random variables and thus may not match the simulation results even when the number of simulation cycles is very large.

This simple example indicates the power and simplicity of the simulation technique. The most significant point is that detailed knowledge of the analytical methods in Chapter 7 is not required to generate the necessary probabilistic information. The task is much simpler if a computer program is used for this purpose.

9.2.5 Extracting Probabilistic Information Using Simulation

The method described in the previous section can also be used to evaluate the risk or reliability of an engineering system. Consider the limit state represented by Equation 7.34 corresponding to a failure mode for a structure. With all the random variables in Equation 7.34 assumed to be statistically independent, the Monte Carlo simulation approach consists of drawing samples of the variables according to their PDFs or PMFs and then feeding them into the mathematical model $g(\ )$. The samples thus obtained would give the probabilistic characteristics of the response random variable Z. The extraction of such information could be cumbersome. However, if the objective is only to estimate the failure probability, that can be done quite simply, as follows.

It is known that if the value of $g(\ )$ is less than zero, it indicates failure. Let N_f be the number of simulation cycles when $g(\ )$ is less than zero and let N be the total number of simulation cycles. Therefore, an estimate of the probability of failure can be expressed as

$$p_f = \frac{N_f}{N}. \tag{9.10}$$

9.2.6 Accuracy and Efficiency of Simulation

The ability of Equation 9.10 to accurately estimate the probability of failure is a matter of concern. Obviously, the accuracy of the estimate will depend on the number of simulation cycles. For a small failure probability and/or small N, the estimate of p_f given by Equation 9.10 may be subject to considerable error. The estimate of the probability of failure would approach the true value as N approaches infinity. The accuracy of Equation 9.10 can be studied in several ways. One way would be to evaluate the variance or COV of the estimated probability of failure (Ayyub and Haldar, 1985). The variance or COV can be estimated by assuming each simulation cycle to constitute a Bernoulli trial, and the number of failures in N trials can be considered to follow a binomial distribution. Then, the COV of p_f can be expressed as

$$\mathrm{COV}(p_f) = \delta_{p_f} \approx \frac{\sqrt{\dfrac{(1-p_f)p_f}{N}}}{p_f}. \tag{9.11}$$

A smaller value of δ_{p_f} is desirable. Equation 9.11 indicates that δ_{p_f} approaches zero as N approaches infinity.

Another way to study the error associated with the number of simulation cycles is by approximating the binomial distribution with a normal distribution and estimating the 95% confidence interval of the estimated probability of failure (Shooman, 1968). It can be shown that

$$P\left[-2\sqrt{\frac{(1-p_f^T)p_f^T}{N}} < \frac{N_f}{N} - p_f^T < 2\sqrt{\frac{(1-p_f^T)p_f^T}{N}}\right] = 0.95 \tag{9.12}$$

where p_f^T is the true probability of failure. The percentage error can be defined as

$$\varepsilon\% = \frac{\dfrac{N_f}{N} - p_f^T}{p_f^T} \times 100\%. \tag{9.13}$$

Combining Equations 9.12 and 9.13, we obtain

$$\varepsilon\% = \sqrt{\frac{(1 - p_f^T)}{N \times p_f^T}} \times 200\%. \tag{9.14}$$

Equation 9.14 indicates that there will be about 20% error if p_f^T is 0.01 and if 10,000 trials were used in the simulation. It can also be stated that there is 95% probability that the probability of failure will be in the range of 0.01 ± 0.002 with 10,000 simulations. Conversely, if the desired error is 10% and p_f^T is 0.01, then from Equation 9.14, the required number of simulations $N = 39{,}600$.

Both Equations 9.11 and 9.14 indicate that the number of simulation cycles to achieve a certain level of accuracy depends on the unknown probability of failure. In many engineering problems, the probability of failure could be smaller than 10^{-5}. Therefore, on average, only 1 out of 100,000 trials would show a failure. Thus, at least 100,000 simulation cycles are required to predict this behavior. For a reliable estimate, at least 10 times this minimum (i.e., 1 million simulation cycles) is usually recommended. If the problem has n random variables, then n million random numbers are necessary if the Monte Carlo simulation is to successfully estimate the probability of failure.

EXAMPLE 9.4

The probability of failure of a beam is under consideration. The following performance function for the beam can be used:

$$g(\ ) = F_y Z - M \tag{9.15}$$

where F_y is the yield stress, Z is the section modulus, and M is the applied bending moment. They are considered to be random variables. Their statistical characteristics are given in Table 9.4.

First, using the MVFOSM method and Equation 9.15, we estimate the safety index to be 3.251 as shown in Table 9.5. When the FORM method is used, the safety index is found to be 2.340. As expected, they are quite different since the limit state equation is nonlinear and the distributions of two out of three random variables are nonnormal. The difference tends to be larger when the difference between the mean values (required for the MVFOSM method) and the design point on the failure surface

Table 9.4 Probabilistic Characteristics of Basic Parameters

Parameter	Mean value	COV	Probability distribution
F_y	38 ksi	0.10	Normal
Z	60 in.^{3}	0.05	Lognormal
M	1,000 kip-in.	0.30	Type II

Table 9.5 Results of MVFOSM and FORM Methods

Reliability methods	β	$p_f = 1 - \Phi(\beta)$
MVFOSM	3.251	0.000577
FORM	2.340	0.009650

(required for the FORM method) gets larger. Thus, MVFOSM may or may not be a realistic measure of probability of failure, depending on the nature of the problem under consideration.

The result obtained by FORM is more reliable and the corresponding probability of failure is found to be 0.009650, a comparatively large number considering actual design practice. This large probability of failure is intentionally considered in this example so that the results can be compared with a simulation study with a reasonable number of simulations. For this example, the probability of failure is of the order of 10^{-2}; thus about 1,000 simulation cycles are necessary for a reasonable estimate of the probability of failure.

Using the limit state given by Equation 9.15 and the Monte Carlo simulation technique, we can calculate the probability of failure of the beam in two different ways: by considering the statistics of the limit state equation, and by counting the failures in different cycles of simulations. In the first method, using the mean and the standard deviation of $g(\ )$, we calculate the safety indices as shown in Column 2 in Table 9.6. Assuming $g(\ )$ is a normal random variable, the corresponding probability of failure is shown in Column 3. As the number of simulation cycles increases, the probability of failure does not show any convergence behavior.

Using the same simulation data with the counting technique given by Equation 9.10, we again calculate the probability of failure of the beam, and the results are shown in Column 4 of Table 9.6. In this case, when the number of simulation cycles is relatively small, such as less than 100, none of the trials resulted in failure of the beam [$g(\ ) < 0$], giving the corresponding probability of failure to be zero. However, as the N values increase, the probability of failure converges to about 0.011. The probabilities of failure obtained by the two methods, that is, the sample statistics and the counting methods, are quite different. If the performance function $g(\ )$ is not normally distributed, this type of difference is expected. In general, since $g(\ )$ may not be normal in most cases, direct simulation with failure counting is superior to the first method.

The probabilities of failure obtained using FORM and the Monte Carlo simulation method are almost the same when the number of simulation cycles is relatively large.

Table 9.6 Summary of Simulation Results

Number of cycles, N	Direct simulation			Conditional expectation VRT		Conditional expectation VRT + antithetic variates VRT	
	Using sample statistics		Counting				
	$\beta = \frac{\mu_g}{\sigma_g}$	$p_f = 1 - \Phi(\beta)$	$p_f = N_f/N$	p_f	COV(p_f)	p_f	COV(p_f)
1	2	3	4	5	6	7	8
10	3.345	0.000412	0.000000*	0.010924	0.204	0.010067	0.048
50	3.224	0.000632	0.000000*	0.009504	0.090	0.010322	0.029
100	3.427	0.000305	0.000000*	0.009368	0.062	0.010327	0.024
250	3.177	0.000744	0.008000	0.010067	0.041	0.010299	0.017
500	2.964	0.001518	0.016000	0.010471	0.030	0.010323	0.012
1000	3.081	0.001032	0.011000	0.010170	0.021	0.010365	0.009

*$N_f = 0$ for these cases.

This indicates that the Monte Carlo method can also be used to evaluate the probability of failure if the number of simulation cycles is relatively large. This also points out the weakness of the direct Monte Carlo simulation technique. The probability of failure of a complicated system is not known in advance; thus, it will be difficult to estimate a reasonable number of simulation cycles in advance. A trial-and-error approach may need to be employed. Also, if the probability of failure is relatively small, as is expected in many engineering designs, the number of cycles necessary to estimate the probability of failure with reasonable accuracy will be very large, making the simulation method time-consuming. With advancements in computer technology, the time required to complete such a large number of simulations may not be a problem, but it could still be prohibitive if the deterministic system analysis for each simulation is computationally intensive.

9.3 VARIANCE REDUCTION TECHNIQUES

The concept behind simulation appears to be simple; however, its application in engineering reliability analysis and its acceptance as an alternative reliability evaluation method depend mainly on the efficiency of the simulation. To achieve efficiency, the number of simulation cycles needs to be greatly reduced. It is the simulator's task to increase the efficiency of the simulation by expediting the execution and minimizing computer storage requirements. Alternatively, efficiency can be increased by reducing the variance or the error of the estimated output variable without disturbing the expected or mean value and without increasing the sample size. This need led to the development of several *variance-reduction techniques* (VRTs). The type of VRT that can be used depends on the particular model under consideration. It is usually impossible to know beforehand how much variance reduction might be achieved using a given technique.

The VRTs can be grouped in several ways. One method is to consider whether the variance reduction method alters the experiment by altering the input scheme, by altering the model, or by special analysis of the output. The VRTs can also be grouped according to description or purpose (i.e., sampling methods, correlation methods, and special methods). These groupings are somewhat arbitrary; however, they produce a better understanding of the concept involved.

The sampling methods either constrain the sample to be representative or distort the sample to emphasize the important aspects of the function being estimated. Some of the commonly used sampling methods are systematic sampling, importance sampling, stratified sampling, Latin hypercube sampling, adaptive sampling, randomization sampling, and conditional expectation. The correlation methods employ strategies to achieve correlation (both positive and negative) between functions, random observations, or different simulations to improve the accuracy of the estimators. Some of the commonly used correlation methods are common random numbers, antithetic variates, and control variates. Other special VRTs available are partition of the region, random quadratic method, biased estimator, and indirect estimator. The VRTs can also be combined to further increase the efficiency of the simulation.

VRTs increase the efficiency and accuracy of the risk or reliability estimation using a relatively small number of simulation cycles; however, they increase the computational difficulty for each simulation, and a considerable amount of expertise may be necessary to implement them. The most desirable feature of simulation, its basic simplicity, is thus lost.

Also, in the age of high-speed computers, the number of simulation cycles or time required to analyze a problem may be less important than in the past. In any case, it is important to understand the logic and concepts behind some of the VRTs commonly used in engineering. The commonly used VRTs include sampling methods and correlation methods. The two types of methods can also be combined, as discussed briefly in the following section.

9.3.1 VRTs in Sampling Methods

9.3.1.1 Importance Sampling

The basic idea of importance sampling is to concentrate the distribution of sampling points in the region of most importance, that is, the area that mainly contributes to the failure probability, instead of spreading them out evenly among the whole range of possible values of the basic variables. One method to achieve this is illustrated in Figure 9.3 (Harbitz, 1986). Efficient variance reduction is obtained by simulating only outside the β-sphere, because no failure occurs within the β-sphere.

The basic mathematical definition of failure probability is given by Equation 7.35, in which the joint probability density function for basic random variables, $\mathbf{X}$, is integrated over the failure region, that is, $g(\mathbf{x}) < 0$. A simple indicator function with respect to the performance function $g(\mathbf{x})$ can be defined as

$$I_g(\mathbf{x}) = \begin{cases} 0, & \text{if } g(\mathbf{x}) > 0 \\ 1, & \text{if } g(\mathbf{x}) < 0 \end{cases} \tag{9.16}$$

With this, Equation 7.35 can be rewritten as

$$p_f = \int_{g(\mathbf{x})<0} I_g(\mathbf{x}) f_X(\mathbf{x}) d\mathbf{x}. \tag{9.17}$$

This is just the expected value of $I_g(\mathbf{x})$. Therefore, in basic Monte Carlo simulation, the probability of failure is simply

$$p_f = \frac{1}{N} \sum_{i=1}^{N} I_g(\mathbf{x}_i) = \frac{n_f}{N} \tag{9.18}$$

where n_f is the number of failures.

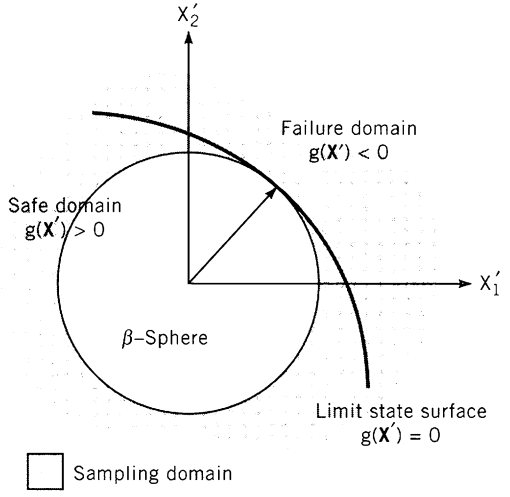


Figure 9.3 Harbitz's Importance Sampling Method

For importance sampling, a new sampling PDF $f_1(\mathbf{x})$ is defined so as to obtain samples in the desired region. $f_1(\mathbf{x})$ is known as the sampling density function. The probability of failure is given by

$$p_f = \int_{g<0} \left[I_g(\mathbf{x}) \frac{f(\mathbf{x})}{f_1(\mathbf{x})} \right] f_1(\mathbf{x}) d\mathbf{x}. \tag{9.19}$$

Equation 9.19 is obtained by multiplying and dividing the integrand in Equation 9.17 by $f_1(\mathbf{x})$. Similar to Equation 9.18, the simulated estimate of p_f is

$$p_f = \frac{1}{N} \sum_{i=1}^{N} I_g(\mathbf{x}_i) \frac{f_{\mathbf{X}}(\mathbf{x}_i)}{f_1(\mathbf{x}_i)} \tag{9.20}$$

where N is the number of simulations and $\mathbf{x}_i$, that is, $(x_1, x_2, ..., x_n)_i$, is the set of values of the basic random variables at the ith simulation.

The accuracy of the importance sampling estimate depends on the choice of the sampling density $f_1(\mathbf{x})$. Several methods have been developed for this purpose (Harbitz, 1986; Karamchandani et al.,1989; Melchers, 1989 etc.)

9.3.1.2 Stratified Sampling

In the stratified sampling method, the domain of integration is divided into several regions, so that emphasis can be placed by simulating more from the regions that contribute to the failure event. The total domain of integration is divided into m mutually exclusive regions, that is, $R_1, R_2, ..., R_m$. Using the theorem of total probability, we can estimate the probability of failure as

$$p_f = \sum_{j=1}^{m} \left[P(R_j) \frac{1}{N_j} \sum_{i=1}^{N_j} I_g(\mathbf{x}_i) \right] \tag{9.21}$$

where $P(R_j)$ is the probability of region R_j, N_j is the number of simulation cycles performed in region R_j, and $I_g(\,)$ is the indicator function defined in Equation 9.16. This strategy ensures that no region is missed.

9.3.1.3 Adaptive Sampling

In the importance sampling method, the region of importance is usually not known in advance. However, the efficiency of the simulation can be improved if it can be updated using the information obtained from the first few simulation cycles. This observation led to the development of the adaptive sampling methods. Two methods of adaptive sampling are multimodal sampling (Karamchandani et al.,1989) and curvature-based sampling (Wu, 1992). Both methods start with a sampling density centered around the design point identified by FORM. As the simulation progresses, the sampling density is updated. In the multimodal method, the sampling density is a weighted sum of density functions centered in different regions. Mahadevan and Dey (1997) used this concept for system reliability analysis. In the curvature-based method, the curvature of the limit state is updated every few simulations, and this information is used in selecting the future samples. It has been reported in the literature (Ayyub and McCuen, 1997; Mahadevan and Dey, 1997) that this technique can reasonably estimate the probability of failure with only 100 to 400 cycles, whereas similar results can only be obtained by using several million direct simulation cycles.

9.3.1.4 Conditional Expectation

In the conditional expectation method, all the basic random variables are simulated except one, known as the *control variable*. The control variable is generally selected to be the random variable with the largest variability, and its CDF provides a known function in terms of other simulated variables. The conditional expectation can be evaluated by this known expression. The only limitation is that the control random variable must be statistically independent of all other random variables in the performance function of interest. The variance is reduced by removing the random fluctuations of the control variable on which conditioning was not performed by not generating them. The steps involved in the conditional expectation method can be summarized as follows:

- Step 1. Select the control variable (i.e., the random variable of the largest variability). Let it be X_j.
- Step 2. Rewrite the quantity to be estimated, in this case p_f, as

$$p_f = P[X_j < g'(X_1, X_2, \ldots, X_{j-1}, X_{j-1}, \ldots, X_n)]. \tag{9.22}$$

- Step 3. Randomly generate values of all these variables except X_j using Monte Carlo simulation and calculate a sample of p_{f_i} as follows:

$$p_{f_i} = P[X_j < g'(x_{1i}, x_{2i}, \ldots, x_{(j-1)i}, x_{(j+1)i}, \ldots, x_{ni})] \tag{9.23}$$

or

$$p_{f_i} = F_{X_j}[g'(x_{1i}, x_{2i}, \ldots, x_{(j-1)i}, x_{(j+1)i}, \ldots, x_{ni})] \tag{9.24}$$

where $F_{X_j}[\ ]$ is the CDF of X_j.

- Step 4. Repeat Step 3 N times then calculate the mean of p_{f_i} denoted as p_f as

$$p_f = \frac{\sum_{i=1}^{N} p_{f_i}}{N}. \tag{9.25}$$

The variance of p_f is

$$\mathrm{Var}(p_f) = \frac{\mathrm{Var}(p_{f_i})}{N} = \frac{\dfrac{1}{N-1}\sum_{i=1}^{N}(p_{f_i} - p_f)^2}{N}. \tag{9.26}$$

EXAMPLE 9.5

Consider Example 9.4, for which the performance function is given by Equation 9.15. In this example, the applied bending moment has the largest variability. To estimate the probability of failure of the beam using the conditional expectation VRT using Equation 9.24, Equation 9.15 needs to be modified as follows:

$$p_f = P(g < 0) = P(M > F_y Z). \tag{9.27}$$

Since M is considered to have a Type II extreme value distribution, its CDF is given by Equation 4.45, and Equation 9.27 can be expressed as

$$P(M > F_y Z) = 1 - \exp\left[-\left(\frac{u}{F_y Z}\right)^k\right] \tag{9.28}$$

where u and k are the parameters of the distribution and can be estimated from the information on the mean and variance of M, using Equations 4.47a and 4.47c. For this example they are [1,000/Γ(0.75) = 1000/1.22542 = 816.05] and 4, respectively. Therefore, the simulation algorithm is to generate F_y and Z as in direct Monte Carlo simulation and use Equation 9.28 to calculate the probability of failure of the beam N times.

The results are summarized in Columns 5 and 6 of Table 9.6. The mean value of the probability of failure of the beam obtained using only 10 cycles with the conditional expectation VRT is very similar to that of the direct Monte Carlo simulation technique. If a larger number of simulation cycles is used for the conditional expectation VRT, the estimates of the mean probability of failure do not change significantly; however, the coefficient of variation of the mean probability of failure decreases, indicating a better estimate.

9.3.2 Correlation-Based VRTs

9.3.2.1 Antithetic Variates

In the antithetic variates method, a negative correlation is induced between different cycles of simulation to decrease the variance of the estimated mean value. If U is a uniformly distributed random variable between 0 and 1 used in the first run of generating a basic random variable X_i, then another random variable $1 - U$ can be introduced in the second run. If U is uniformly distributed between 0 and 1, then $1 - U$ is also uniformly distributed in the same range and is negatively correlated to U. The concept can be used very efficiently when it is combined with other VRTs, for example, the conditional expectation VRT discussed in the previous section. It is elaborated further in the following section.

9.3.3 Combined Conditional Expectation and Antithetic Variates Method

As discussed in the previous sections, VRTs can be used individually to improve the overall efficiency of the simulation. Using some of these VRTs together also will further improve efficiency. One simple combination would be the conditional expectation plus antithetic variates VRT. The following steps can be followed to combine the two VRTs.

- Step 1. Select the control variable and define p_f in terms of its CDF as in the conditional expectation VRT. Refer to Equation 9.27 as an example.
- Step 2. Using the Monte Carlo simulation method, generate random numbers U_i's for all the basic random variables except the control variable; using Equation 9.24, calculate the corresponding probability of failure, denoted as $p_{f_i}^{(1)}$.
- Step 3. Use another set of random numbers corresponding to $(1 - U_i)$'s and, following the procedures in Step 2, calculate $p_{f_i}^{(2)}$.
- Step 4. Evaluate the average value of the two runs in Steps 2 and 3 as

$$p_{f_i} = \frac{p_{f_i}^{(1)} + p_{f_i}^{(2)}}{2}. \tag{9.29}$$

- Step 5. Repeat Steps 2 to 4 N times. An estimate of the mean and variance of p_f is given by

$$p_f = \frac{\sum_{i=1}^{N} p_{f_i}}{N} \tag{9.30}$$

and

$$\text{Var}(p_f) = \frac{\text{Var}(p_{f_i})}{N} = \frac{\text{Var}(p_{f_i}^{(1)}) + \text{Var}(p_{f_i}^{(2)}) + 2\text{Cov}(p_{f_i}^{(1)}, p_{f_i}^{(2)})}{4N}. \tag{9.31}$$

Since the covariance of $p_{f_i}^{(1)}$ and $p_{f_i}^{(2)}$ is negative, the variance of p_f will be reduced.

EXAMPLE 9.6

Example 9.5 is considered again. The conditional expectation and antithetic variates VRTs are combined to calculate the probabilities of failure of the same beam. The results are shown in Columns 7 and 8 of Table 9.6. Comparing Columns 5 and 7 reveals that the mean value of the probability of failure did not change considerably. However, the coefficient of variation of its estimation is considerably smaller for the combined VRTs than for the conditional expectation VRT alone. Thus, the conditional expectation plus antithetic variates VRT is more efficient than the conditional expectation VRT alone, for this example.

In this example, about 1,000 direct simulation cycles gave a reasonable estimate of the probability of failure. However, if the probability of failure is about 10^{-5}, as is expected in most structural engineering problems, the number of direct simulation cycles is expected to be about a million. Thus, whenever possible, some type of VRT should be used to increase the efficiency of simulation.

The example also indicates the important role of simulation in reliability estimation of engineering systems. As mentioned earlier, simulation is used to verify results obtained using more sophisticated analytical methods. The latter can only be used by experts in the areas of risk and reliability, whereas the simulation method is quite simple to use. Also, the simulation method is robust. It can provide failure probability estimates for any problem. The analytical methods might sometimes fail to converge in their iterations.

9.4 SIMULATION OF CORRELATED RANDOM VARIABLES

The discussion in the previous sections assumes that all the random variables are uncorrelated. In some cases, it may be necessary to estimate the probability of failure of a structure when some or all the random variables are correlated.

The fundamental concepts that need consideration are how to convert the correlated random variables to uncorrelated or statistically independent random variables, and how to modify the original function expressed in terms of correlated variables into a function of uncorrelated random variables.

The methods proposed by Morgenstern (1956) and Nataf (1962) can be used to convert correlated variables to uncorrelated random variables. Nataf's model is discussed

very briefly here, since it is more flexible than Morgenstern's model. Suppose X_1, X_2, ..., X_n are correlated random variables with covariance matrix [C] given by Equation 8.8. The correlation coefficient between X_i and X_j is denoted as ρ_{X_i,X_j}. The X_i's can be transformed into standard normal variates U_i's as

$$U_i = \Phi^{-1}\left[F_{X_i}(x_i)\right]; \quad i = 1,2,\ldots,n. \tag{9.32}$$

The U_i's have a zero mean and unit standard deviation. However, the transformation may change the correlation coefficient between any two correlated random variables to ρ'_{U_1,U_2}, or simply ρ' in subsequent discussions. Using Equations 3.36 and 3.39, we can show that

$$\rho_{X_1,X_2}\sigma_{X_1}\sigma_{X_2} = E(X_1X_2) - E(X_1)E(X_2). \tag{9.33}$$

With the transformation of X_i's to U_i's, the expectation operation in Equation 9.33 becomes difficult, and in some cases, approximate solutions such as the use of Taylor's series may be necessary. Nataf's model can be used, but the calculations become very tedious. Der Kiureghian and Liu (1985) suggested an empirical relationship between the two correlation coefficients as

$$\rho' = F\rho_{X_i,X_j} \tag{9.34}$$

in which $F \geq 1.0$. Liu and Der Kiureghian (1986) estimated the values of F for several two-parameter distributions of X_i and X_j, as shown in Table 9.7. If one of the two variables is normal, F may be a constant or a function of the COV of the other random variable. However, in all cases, the maximum error in the estimation of F is very small. For combinations of distributions not shown in Table 9.7, refer to Liu and Der Kiureghian (1986).

The symbol ρ' indicates the correlation coefficient between two standard normal variables with zero mean and unit standard deviation. The correlation matrix for this case can be written as

$$[\mathbf{C}_U] = \begin{bmatrix} 1 & \rho' \\ \rho' & 1 \end{bmatrix}. \tag{9.35}$$

Table 9.7 Evaluation of F Parameter

X_i	X_j	F	Maximum error
Normal	Normal	1.0	
Normal	Uniform	1.023	0.0%
Normal	Shifted exponential	1.107	0.0%
Normal	Shifted Rayleigh	1.014	0.0%
Normal	Type I largest value	1.031	0.0%
Normal	Type I smallest value	1.031	0.0%
Normal	Lognormal	$\delta_{X_j} / \sqrt{\ln(1+\delta_{X_j}^2)}$	Exact
Normal	Gamma	$1.001 - 0.007\delta_{X_j} + 0.118\delta_{X_j}^2$	0.0%
Normal	Type II largest value	$1.030 + 0.238\delta_{X_j} + 0.364\delta_{X_j}^2$	0.1%
Normal	Type III smallest value	$1.031 - 0.195\delta_{X_j} + 0.328\delta_{X_j}^2$	0.1%

Denoting **V** as the uncorrelated standard normal variables, we can show that

$$\{\mathbf{U}\} = [\mathbf{T}]\{\mathbf{V}\} \tag{9.36}$$

As shown in Section 8.3, the transformation matrix **T** is composed of the eigenvectors of $[\mathbf{C}_U]$ and can be shown to be

$$[\mathbf{T}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{9.37}$$

The corresponding eigenvectors $(1 - \rho)$ and $(1 + \rho)$ are the variances of V_1 and V_2. Thus, V_1 and V_2 are two independent normal variables with zero mean, and the corresponding variances are $(1 - \rho)$ and $(1 + \rho)$, respectively.

The modifications of the original function to be simulated in terms of V_i's are discussed in the following sections with the help of examples. For ease of presentation, the discussion is divided into two parts: simulation of correlated normal random variables, and simulation of correlated nonnormal random variables.

9.4.1 Simulation of Correlated Normal Variables

To demonstrate the simulation procedure for correlated random variables, the example discussed in Section 9.2.1 can again be considered. For this example, the expression for the design bending moment at the midspan of the beam is given by Equation 9.1. Previously, the two random variables W and P in Equation 9.1 were considered to be statistically independent. The task now is to simulate them and calculate the bending moments if they are correlated.

For the purpose of illustration, first consider both W and P to be normal random variables with means of 2 kip/ft and 15 kip, respectively, and corresponding standard deviations of 0.2 kip/ft and 2.5 kip, respectively. The correlation coefficient of the two variables is assumed to be $\rho_{W,P} = 0.3$.

To demonstrate the difference in simulation between uncorrelated and correlated random variables, W and P are first considered to be independent normal random variables. Using 10 simulation cycles, we can calculate the bending moments at the midspan of the beam, with results summarized in Table 9.8. In this example, the first 10 uniform numbers between 0 and 1 are assigned to W, and the next 10 uniform numbers between 0 and 1 are assigned to P. The mean and the standard deviation of the bending moment are calculated to be 344.29 kip-ft and 26.34 kip-ft, respectively.

To consider the effect of correlation between W and P, the following steps can be performed.

- Step 1. Transform the correlated normal variables to correlated standard normal variables, U_i's. The U_i's have a zero mean and unit standard deviation. For this particular example, it can be shown that

$$U_1 = \frac{W - \mu_W}{\sigma_W}$$

or

$$W = \mu_W + \sigma_W U_1.$$

Table 9.8 Monte Carlo Simulations for Uncorrelated Normal Variables
$W \sim N(2, 0.2)$, $P \sim N(15, 2.5)$, $\rho_{WP} = 0$

$W \sim N(2, 0.2)$			$P \sim N(15, 2.5)$			$m_i = 112.5\, w_i + 7.5\, p_i$
u_i	s_i	w_i	u_i	s_i	p_i	m_i
0.86061	1.08306	2.21661	0.83771	0.98509	17.46273	380.33910
0.92546	1.44279	2.28856	0.73006	0.61299	16.53248	381.45660
0.41806	−0.20686	1.95863	0.56341	0.15962	15.39905	335.83875
0.28964	−0.55444	1.88911	0.82178	0.92217	17.30543	342.31560
0.14225	−1.07027	1.78595	0.32715	−0.44780	13.88050	305.02313
0.44961	−0.12665	1.97467	0.68853	0.49169	16.22923	343.86960
0.24653	−0.68545	1.86291	0.74358	0.65442	16.63605	334.34775
0.21687	−0.78281	1.84344	0.24672	−0.68485	13.28788	307.04610
0.56503	0.16373	2.03275	0.90324	1.30024	18.25060	365.56388
0.40015	−0.25296	1.94941	0.79263	0.81558	17.03895	347.10075

Mean of moment = 344.29 kip-ft; standard deviation of moment = 26.34 kip-ft.

Similarly,

$$P = \mu_P + \sigma_P U_2.$$

- Step 2. Evaluate the correlation coefficient ρ' between U_1 and U_2 in terms of the correlation coefficient of the original correlated normal variables. As shown in Table 9.7, when both variables are normal, the parameter F is 1.0. Thus,

$$\rho' = \rho_{W,P} = 0.3.$$

- Step 3. Transform the correlated standard normal variables U_i's to uncorrelated standard normal variables V_i's. Using Equation 9.36, we can show that

$$U_1 = \frac{1}{\sqrt{2}}(V_1 + V_2)$$

and

$$U_2 = \frac{1}{\sqrt{2}}(-V_1 + V_2).$$

- Step 4. Express the function to be simulated in terms of V_i's. For the problem under consideration, it can be shown that

$$W = \mu_W + \frac{\sigma_W}{\sqrt{2}}(V_1 + V_2)$$

and

$$P = \mu_P + \frac{\sigma_P}{\sqrt{2}}(-V_1 + V_2).$$

Thus,

$$M = 112.5W + 7.5P = 112.5\left[\mu_W + \frac{\sigma_W}{\sqrt{2}}(V_1 + V_2)\right]$$

$$+7.5\left[\mu_P + \frac{\sigma_P}{\sqrt{2}}(-V_1 + V_2)\right].$$

By substituting the mean and standard deviation values in the equation and simplifying, we can show that

$$M = 337.5 + 2.65165V_1 + 29.16815V_2.$$

As mentioned earlier, V_1 and V_2 are normal random variables with zero mean and corresponding standard deviations of $\sqrt{1-\rho'} = \sqrt{1-0.3} = 0.837$ and $\sqrt{1+\rho'} = \sqrt{1+0.3} = 1.140$, respectively.

- Step 5. Carry out standard Monte Carlo simulation using the modified function expressed in terms of V_i's. Considering the first 10 uniform random numbers between 0 and 1 in Table 9.1 for V_1 and the next 10 for V_2, we can calculate the bending moments as summarized in Table 9.9. The mean and standard deviation of the bending moment for the correlated case are 353.29 kip-ft and 22.03 kip-ft, respectively.

9.4.2 Simulation of Correlated Nonnormal Variables

In the previous example, both W and P could have nonnormal distributions. The general steps discussed next can be used for the simulation of correlated nonnormal variables. For simplicity of discussion, consider W to be a normal random variable with a mean of 2.0 kip/ft and a standard deviation of 0.2 kip/ft. However, P is a lognormal random variable with a mean of 15 kip and a standard deviation of 2.5 kip. Equations 4.10 and 4.11 are used to find the two parameters of the lognormal distribution: $\lambda_P = 2.694$ and $\zeta_P = 0.166$. The coefficient of variation of P is $\delta_P = 0.167$. Again, the five steps discussed in the previous section can be carried out in the following way.

- Step 1. Since W is a normal random variable, it can be shown that

$$W = \mu_W + \sigma_W U_1.$$

Since P is a lognormal random variable, the following additional calculations are necessary:

Table 9.9 Monte Carlo Simulations for Correlated Normal Variables

$W \sim N(2, 0.2)$, $P \sim N(15, 2.5)$, $\rho_{W,P} = 0.3$

$V_1 \sim N(0, 0.837)$			$V_2 \sim N(0, 1.140)$			$m_i = 337.5 + 2.65165\ v_{1_i} + 29.16815\ v_{2_i}$
u_i	s_i	v_{1_i}	u_i	s_i	v_{2_i}	m_i
0.86061	1.08306	0.90652	0.83771	0.98509	1.12300	372.65961
0.92546	1.44279	1.20762	0.73006	0.61299	0.69881	361.08518
0.41806	−0.20686	−0.17314	0.56341	0.15962	0.18197	342.34862
0.28964	−0.55444	−0.46407	0.82178	0.92217	1.05127	366.93305
0.14225	−1.07027	−0.89582	0.32715	−0.44780	−0.51049	320.23455
0.44961	−0.12665	−0.10601	0.68853	0.49169	0.56053	353.56852
0.24653	−0.68545	−0.57372	0.74358	0.65442	0.74604	357.73930
0.21687	−0.78281	−0.65521	0.24672	−0.68485	−0.78073	312.99016
0.56503	0.16373	0.13704	0.90324	1.30024	1.48227	381.09846
0.40015	−0.25296	−0.21173	0.79263	0.81558	0.92976	364.05795

Mean of moment = 353.29 kip-ft; standard deviation of moment = 22.03 kip-ft.

$$\Phi(U_2) = \Phi\left(\frac{\ln P - \lambda_P}{\zeta_P}\right)$$

or

$$U_2 = \frac{\ln P - \lambda_P}{\zeta_P}, \quad \text{or} \quad \ln P = \lambda_P + U_2\zeta_P$$

or

$$P = \exp(\lambda_P + U_2\zeta_P).$$

- Step 2. From Table 9.7, the parameter F can be estimated as

$$F = \frac{\delta_P}{\sqrt{\ln(1+\delta_P^2)}} = \frac{0.167}{\sqrt{\ln(1+0.167^2)}} = 1.00692.$$

Using Equation 9.34,

$$\rho' = F\rho_{W,P} = 1.00692 \times 0.3 = 0.302.$$

- Step 3. The relationships between the U_i's and V_i's discussed in the previous example will remain the same.
- Step 4. For the problem under consideration, it can be shown that

$$W = \mu_W + \frac{\sigma_W}{\sqrt{2}}(V_1 + V_2) = 2.0 + \frac{0.2}{\sqrt{2}}(V_1 + V_2)$$

and

$$P = \exp\left[\lambda_P + \frac{\zeta_P}{\sqrt{2}}(-V_1 + V_2)\right] = \exp\left[2.694 + \frac{0.166}{\sqrt{2}}(-V_1 + V_2)\right].$$

Thus, the function to estimate the bending moment can be expressed in terms of V_i's. As discussed before, V_1 and V_2 are normal random variables with zero mean and corresponding standard deviations of $\sqrt{1-\rho'} = \sqrt{1-0.302} = 0.835$ and $\sqrt{1+\rho'} = \sqrt{1+0.302} = 1.141$.

- Step 5. Considering the first 10 uniform numbers between 0 and 1 in Table 9.1 for V_1 and the next 10 for V_2, we can calculate the bending moments, as summarized in Table 9.10. The mean value of the bending moment is found to be 352.22 kip-ft, and the corresponding standard deviation is 22.38 kip-ft.

9.5 CONCLUDING REMARKS

The use of simulation techniques to estimate the probability of failure for both explicit and implicit performance functions is discussed in this chapter. The simulation method can be carried out without knowing more complicated analytical techniques and with a working knowledge of probability and statistics. This method is also robust. The simulation method can provide estimates for any problem, whereas analytical methods may not always converge in their iterations. With the advancement in computer technology, simulation is becoming an attractive alternative to classical analytical methods.

The accuracy of simulation is always a major concern. Simulation is expected to give accurate results as the number of simulation trials approaches infinity. In the age of high-speed computing, a large number of simulation cycles may not be an important hurdle. However, it is important to understand the logic and concepts behind some of the commonly used variance reduction techniques to increase the efficiency and accuracy of simulation. Some of them are introduced in this chapter.

Table 9.10 Monte Carlo Simulations for Correlated Nonnormal Variables
$W \sim N(2, 0.2)$, $P \sim LN(2.694, 0.166)$, $\rho_{W,P} = 0.3$

$$m_i = 112.5w_i + 7.5p_i$$

$$w_i = 2 + \frac{0.2}{\sqrt{2}}(v_{1_i} + v_{2_i})$$

$$p_i = \exp\left[2.694 + \frac{0.166}{\sqrt{2}}(-v_{1_i} + v_{2_i})\right]$$

$V_1 \sim N(0, 0.835)$			$V_2 \sim N(0, 1.141)$			
u_i	s_i	v_{1_i}	u_i	s_i	v_{2_i}	m_i
0.86061	1.08306	0.90436	0.83771	0.98509	1.12399	371.09798
0.92546	1.44279	1.20473	0.73006	0.61299	0.69942	359.83725
0.41806	−0.20686	−0.17273	0.56341	0.15962	0.18213	340.79820
0.28964	−0.55444	−0.46296	0.82178	0.92217	1.05220	366.89708
0.14225	−1.07027	−0.89368	0.32715	−0.44780	−0.51094	318.68070
0.44961	−0.12665	−0.10575	0.68853	0.49169	0.56102	352.20398
0.24653	−0.68545	−0.57235	0.74358	0.65442	0.74669	357.28080
0.21687	−0.78281	−0.65365	0.24672	−0.68485	−0.78141	311.44740
0.56503	0.16373	0.13671	0.90324	1.30024	1.48357	380.70840
0.40015	−0.25296	−0.21122	0.79263	0.81558	0.93058	363.28470

Mean of moment = 352.22 kip-ft.; standard deviation of moment = 22.38 kip-ft.

The basic simulation method assumes that the random variables to be simulated are essentially statistically independent. In practice, however, some or all the random variables may be correlated. Simulation of correlated normal and nonnormal random variables is therefore discussed in this chapter.

A working knowledge of simulation methods is helpful because they are not only simple and robust but also necessary for the validation of more sophisticated analytical methods.

9.6 PROBLEMS

(Note: The following problems are intended for homework assignments using the uniform random numbers given in Table 9.1. Answers for homework assignments are requested with very few simulations, since the number of uniform random numbers in Table 9.1 is very limited and a considerable amount of time would be needed for a large simulation cycle using hand calculation. This way students will get unique results and it will be easier for the teacher to check their accuracy. However, if a teacher prefers to use a computer, then the number of simulations could be large. In that case, the results may vary depending on the computer being used and the seed value used to generate uniform random numbers. However, the results are expected to be similar, particularly when relatively large numbers of simulation cycles are used.)

9.1 The total shear resistance of soil between B and C against slope failure is given by (refer to Figure P6.14 in Chapter 6)

$$F = (C + P\tan\phi)L$$

where C is the cohesion, P is the pressure normal to arc BC, ϕ is the friction angle of soil, and L is the length of arc BC = 10 ft. Assume C, P, and ϕ are statistically independent ran-

dom variables and L is a constant. Further assume that C is a uniformly distributed random variable between 0 and 1, P is uniformly distributed between 1 and 3, and ϕ is uniformly distributed between 20 and 30 degrees.

(a) Calculate the first-order mean and variance of F.

(b) Using 10 cycles of simulation, calculate the mean and variance of F.

9.2 The drag force, F_D, acting on an immersed body by moving fluid can be calculated as

$$F_D = C_D A \frac{\rho U^2}{2}$$

where C_D is the drag coefficient, A is the projected area of the body on a plane normal to the flow, ρ is the mass density of the fluid, and U is the undisturbed velocity of the fluid. Suppose A and ρ are constants with values of 10 ft^2 and 1.94 slug/ft^3, respectively. C_D and U are both assumed to be statistically independent normal random variables with means of 0.5 and 10 ft/s, respectively, and corresponding COVs of 0.1 and 0.2, respectively.

(a) Calculate the first-order mean and variance of F_D.

(b) Using 15 cycles of simulation, calculate the mean and variance of F_D.

9.3 A simply supported beam of span L and stiffness EI is loaded with a concentrated load P at the midspan and a uniformly distributed load w along the length of the beam. The maximum deflection at the midspan can be calculated as

$$\delta_{\max} = \frac{PL^3}{48EI} + \frac{5}{385}\frac{wL^4}{EI}.$$

Suppose L and EI are constants of values 30 ft and 4.495×10^7 kip-in.^{2}, but P is a normal random variable with a mean of 50 kip and a standard deviation of 10 kip, and w is a log-normal variable with a mean of 1 kip/ft and a standard deviation of 0.1 kip/ft. Using 20 cycles of simulation, calculate the mean and standard deviation of the maximum deflection of the beam.

9.4 Consider Problem 6.10. It can be shown that the coefficient of consolidation, c_v, has a log-normal distribution with $\lambda = -11.3267$ and $\zeta = 0.1414$. The probability of c_v being greater than 1.2×10^{-5} m^2/min can be calculated as 0.51197. Using 20 cycles of simulation, calculate the probability of c_v being greater than 1.2×10^{-5} m^2/min. Discuss the accuracy of the result.

9.5 The fully plastic flexural capacity of a steel beam section can be given by YZ, where Y is the yield strength of steel and Z is the plastic section modulus of the section. If the applied bending movement at a section of interest is M, the performance function can be defined as

$$g(\,) = YZ - M.$$

Assume Z is a constant with a value of 40 in.^{3}, Y is uniformly distributed between 25 and 55 ksi, and M is uniformly distributed between 500 and 1,500 kip-in. Assume Y and M are statistically independent. Using 20 cycles of simulation, calculate the probability of failure of the beam. Briefly discuss the accuracy of the result and how it could be improved.

9.6 In Problem 9.5, suppose Y and M are statistically independent normal random variables with mean values of 40 ksi and 1,000 kip-in., respectively, and corresponding COVs of 0.125 and 0.20, respectively. Assume Z is a constant with a value of 40 in.^{3}.

(a) What is the distribution of $g(\,)$ and its parameters?

(b) What is the probability of failure of the beam?

(c) Using 15 cycles of simulation, calculate the probability of failure of the beam.

(d) Using 15 cycles of simulation and the conditional expectation VRT, calculate the probability of failure of the beam.

(e) Using 15 cycles of simulation and the combined conditional expectation and antithetic variates VRT, calculate the probability of failure of the beam.

9.7 In Problem 9.5, if Y and M are statistically independent lognormal random variables and all other information remains the same, calculate the probability of failure of the beam in the following ways:

(a) Using 15 cycles of simulation.

(b) Using 15 cycles of simulation and the conditional expectation VRT.

(c) Using 15 cycles of simulation and the combined conditional expectation and antithetic variates VRT.

(d) Comment on the accuracy of the results in Parts (a), (b), and (c).

9.8 In Problem 9.6, if Y and M are correlated normal variables with $\rho_{Y,M} = 0.70$ and all other information remains the same, calculate the probability of failure of the beam using 15 simulation cycles.

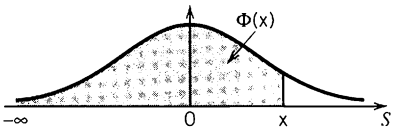
9.9 Repeat Problem 9.8, with Y as a normal variable with a mean of 40 ksi and a COV of 0.125 and M as a uniform random variable between 500 and 1,500 kip-in., and the correlation coefficient between them is still 0.70.

9.10 Repeat Problem 9.8, with Y as a normal variable with a mean of 40 ksi and a COV of 0.125 and M as a lognormal variable with a mean of 1,000 kip-in. and a COV of 0.20, and the correlation coefficient between them still 0.70.

Appendix 1

Table of the CDF of the Standard Normal Distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(s^2/2)} ds$$



x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.50000	0.25	0.59871	0.50	0.69146	0.75	0.77337
0.01	0.50399	0.26	0.60257	0.51	0.69497	0.76	0.77637
0.02	0.50798	0.27	0.60642	0.52	0.69847	0.77	0.77935
0.03	0.51197	0.28	0.61026	0.53	0.70194	0.78	0.78230
0.04	0.51595	0.29	0.61409	0.54	0.70540	0.79	0.78524
0.05	0.51994	0.30	0.61791	0.55	0.70884	0.80	0.78814
0.06	0.52392	0.31	0.62172	0.56	0.71226	0.81	0.79103
0.07	0.52790	0.32	0.62552	0.57	0.71566	0.82	0.79389
0.08	0.53188	0.33	0.62930	0.58	0.71904	0.83	0.79673
0.09	0.53586	0.34	0.63307	0.59	0.72240	0.84	0.79955
0.10	0.53983	0.35	0.63683	0.60	0.72575	0.85	0.80234
0.11	0.54380	0.36	0.64058	0.61	0.72907	0.86	0.80511
0.12	0.54776	0.37	0.64431	0.62	0.73237	0.87	0.80785
0.13	0.55172	0.38	0.64803	0.63	0.73565	0.88	0.81057
0.14	0.55567	0.39	0.65173	0.64	0.73891	0.89	0.81327
0.15	0.55962	0.40	0.65542	0.65	0.74215	0.90	0.81594
0.16	0.56356	0.41	0.65910	0.66	0.74537	0.91	0.81859
0.17	0.56749	0.42	0.66276	0.67	0.74857	0.92	0.82121
0.18	0.57142	0.43	0.66640	0.68	0.75175	0.93	0.82381
0.19	0.57535	0.44	0.67003	0.69	0.75490	0.94	0.82639
0.20	0.57926	0.45	0.67364	0.70	0.75804	0.95	0.82894
0.21	0.58317	0.46	0.67724	0.71	0.76115	0.96	0.83147
0.22	0.58706	0.47	0.68082	0.72	0.76424	0.97	0.83398
0.23	0.59095	0.48	0.68439	0.73	0.76730	0.98	0.83646
0.24	0.59483	0.49	0.68793	0.74	0.77035	0.99	0.83891

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
1.00	0.84134	1.50	0.93319	2.00	0.97725	2.50	0.99379
1.01	0.84375	1.51	0.93440	2.01	0.97778	2.51	0.99396
1.02	0.84614	1.52	0.93574	2.02	0.97831	2.52	0.99413
1.03	0.84849	1.53	0.93699	2.03	0.97882	2.53	0.99430
1.04	0.85083	1.54	0.93822	2.04	0.97932	2.54	0.99446
1.05	0.85314	1.55	0.93943	2.05	0.97982	2.55	0.99461
1.06	0.85543	1.56	0.94062	2.06	0.98030	2.56	0.99477
1.07	0.85769	1.57	0.94179	2.07	0.98077	2.57	0.99492
1.08	0.85993	1.58	0.94295	2.08	0.98124	2.58	0.99506
1.09	0.86214	1.59	0.94408	2.09	0.98169	2.59	0.99520
1.10	0.86433	1.60	0.94520	2.10	0.98214	2.60	0.99534
1.11	0.86650	1.61	0.94630	2.11	0.98257	2.61	0.99547
1.12	0.86864	1.62	0.94738	2.12	0.98300	2.62	0.99560
1.13	0.87076	1.63	0.94845	2.13	0.98341	2.63	0.99573
1.14	0.87286	1.64	0.94950	2.14	0.98382	2.64	0.99585
1.15	0.87493	1.65	0.95053	2.15	0.98422	2.65	0.99598
1.16	0.87698	1.66	0.95154	2.16	0.98461	2.66	0.99609
1.17	0.87900	1.67	0.95254	2.17	0.98500	2.67	0.99621
1.18	0.88100	1.68	0.95352	2.18	0.98537	2.68	0.99632
1.19	0.88298	1.69	0.95449	2.19	0.98574	2.69	0.99643
1.20	0.88493	1.70	0.95543	2.20	0.98610	2.70	0.99653
1.21	0.88686	1.71	0.95637	2.21	0.98645	2.71	0.99664
1.22	0.88877	1.72	0.95728	2.22	0.98679	2.72	0.99674
1.23	0.89065	1.73	0.95818	2.23	0.98713	2.73	0.99683
1.24	0.89251	1.74	0.95907	2.24	0.98745	2.74	0.99693
1.25	0.89435	1.75	0.95994	2.25	0.98778	2.75	0.99702
1.26	0.89617	1.76	0.96080	2.26	0.98809	2.76	0.99711
1.27	0.89796	1.77	0.96164	2.27	0.98840	2.77	0.99720
1.28	0.89973	1.78	0.96246	2.28	0.98870	2.78	0.99728
1.29	0.90147	1.79	0.96327	2.29	0.98899	2.79	0.99736
1.30	0.90320	1.80	0.96407	2.30	0.98928	2.80	0.99744
1.31	0.90490	1.81	0.96485	2.31	0.98956	2.81	0.99752
1.32	0.90658	1.82	0.96562	2.32	0.98983	2.82	0.99760
1.33	0.90824	1.83	0.96638	2.33	0.99010	2.83	0.99767
1.34	0.90988	1.84	0.96712	2.34	0.99036	2.84	0.99774
1.35	0.91149	1.85	0.96784	2.35	0.99061	2.85	0.99781
1.36	0.91308	1.86	0.96856	2.36	0.99086	2.86	0.99788
1.37	0.91466	1.87	0.96926	2.37	0.99111	2.87	0.99795
1.38	0.91621	1.88	0.96995	2.38	0.99134	2.88	0.99801
1.39	0.91774	1.89	0.97062	2.39	0.99158	2.89	0.99807
1.40	0.91924	1.90	0.97128	2.40	0.99180	2.90	0.99813
1.41	0.92073	1.91	0.97193	2.41	0.99202	2.91	0.99819
1.42	0.92220	1.92	0.97257	2.42	0.99224	2.92	0.99825
1.43	0.92364	1.93	0.97320	2.43	0.99245	2.93	0.99831
1.44	0.92507	1.94	0.97381	2.44	0.99266	2.94	0.99836
1.45	0.92647	1.95	0.97441	2.45	0.99286	2.95	0.99841
1.46	0.92785	1.96	0.97500	2.46	0.99305	2.96	0.99846
1.47	0.92922	1.97	0.97558	2.47	0.99324	2.97	0.99851
1.48	0.93056	1.98	0.97615	2.48	0.99343	2.98	0.99856
1.49	0.93189	1.99	0.97670	2.49	0.99361	2.99	0.99861

x	$\Phi(x)$	x	$\Phi(x)$	x	$1-\Phi(x)$
3.00	0.99865	3.50	0.99977	4.00	3.1686e−05
3.01	0.99869	3.51	0.99978	4.05	2.5622e−05
3.02	0.99874	3.52	0.99978	4.10	2.0669e−05
3.03	0.99878	3.53	0.99979	4.15	1.6633e−05
3.04	0.99882	3.54	0.99980	4.20	1.3354e−05
3.05	0.99886	3.55	0.99981	4.25	1.0696e−05
3.06	0.99889	3.56	0.99981	4.30	8.5460e−06
3.07	0.99893	3.57	0.99982	4.35	6.8121e−06
3.08	0.99896	3.58	0.99983	4.40	5.4170e−06
3.09	0.99900	3.59	0.99983	4.45	4.2972e−06
3.10	0.99903	3.60	0.99984	4.50	3.4008e−06
3.11	0.99906	3.61	0.99985	4.55	2.6849e−06
3.12	0.99910	3.62	0.99985	4.60	2.1146e−06
3.13	0.99913	3.63	0.99986	4.65	1.6615e−06
3.14	0.99916	3.64	0.99986	4.70	1.3023e−06
3.15	0.99918	3.65	0.99987	4.75	1.0183e−06
3.16	0.99921	3.66	0.99987	4.80	7.9435e−07
3.17	0.99924	3.67	0.99988	4.85	6.1815e−07
3.18	0.99926	3.68	0.99988	4.90	4.7987e−07
3.19	0.99929	3.69	0.99989	4.95	3.7163e−07
3.20	0.99931	3.70	0.99989	5.00	2.8710e−07
3.21	0.99934	3.71	0.99990	5.10	1.7012e−07
3.22	0.99936	3.72	0.99990	5.20	9.9834e−08
3.23	0.99938	3.73	0.99990	5.30	5.8022e−08
3.24	0.99940	3.74	0.99991	5.40	3.3396e−08
3.25	0.99942	3.75	0.99991	5.50	1.9036e−08
3.26	0.99944	3.76	0.99992	5.60	1.0746e−08
3.27	0.99946	3.77	0.99992	5.70	6.0077e−09
3.28	0.99948	3.78	0.99992	5.80	3.3261e−09
3.29	0.99950	3.79	0.99992	5.90	1.8236e−09
3.30	0.99952	3.80	0.99993	6.00	9.9012e−10
3.31	0.99953	3.81	0.99993	6.10	5.3238e−10
3.32	0.99955	3.82	0.99993	6.20	2.8347e−10
3.33	0.99957	3.83	0.99994	6.30	1.4947e−10
3.34	0.99958	3.84	0.99994	6.40	7.8049e−11
3.35	0.99960	3.85	0.99994	6.50	4.0358e−11
3.36	0.99961	3.86	0.99994	6.60	2.0665e−11
3.37	0.99962	3.87	0.99995	6.70	1.0479e−11
3.38	0.99964	3.88	0.99995	6.80	5.2616e−12
3.39	0.99965	3.89	0.99995	6.90	2.6161e−12
3.40	0.99966	3.90	0.99995	7.00	1.2881e−12
3.41	0.99968	3.91	0.99995	7.10	6.2805e−13
3.42	0.99969	3.92	0.99996	7.20	3.0320e−13
3.43	0.99970	3.93	0.99996	7.30	1.4500e−13
3.44	0.99971	3.94	0.99996	7.40	6.8612e−14
3.45	0.99972	3.95	0.99996	7.50	3.2196e−14
3.46	0.99973	3.96	0.99996	7.60	1.4988e−14
3.47	0.99974	3.97	0.99996	7.70	6.8834e−15
3.48	0.99975	3.98	0.99997	7.80	3.1086e−15
3.49	0.99976	3.99	0.99997	7.90	1.4433e−15

Appendix 2

Evaluation of Gamma Function

- Exact evaluation (Euler's integral)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

- Approximate evaluation of gamma function

1. Integer values, x

   $\Gamma(1 + x) = x\Gamma(x) = x!$

2. Noninteger values, x

   a. For $1 \le z \le 2$ (i.e., $0 \le x \le 1$), $\Gamma(z) = \Gamma(1 + x)$

      Polynomial approximation for gamma function*

      i. $\Gamma(1 + x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \varepsilon(x)$, $|\varepsilon(x)| \le 5 \times 10^{-5}$

      $a_1 = -0.5748646$, $a_2 = 0.9512363$, $a_3 = -0.6998588$
      $a_4 = 0.4245549$, $a_5 = -0.1010678$

      ii. $\Gamma(1 + x) = 1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 + \varepsilon(x)$, $|\varepsilon(x)| \le 3 \times 10^{-7}$

      $b_1 = -0.577191652$, $b_2 = 0.988205891$, $b_3 = -0.897056937$,
      $b_4 = 0.918206857$, $b_5 = -0.756704078$, $b_6 = 0.482199394$,
      $b_7 = -0.193527818$, $b_8 = 0.035868343$

*Abramowitz, M., and Stegun, I.A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series 55, Washington, DC, 1964.

Example: $\Gamma(11/8) = \Gamma(1 + 3/8) = 1 - 0.577191652 \times (3/8) + 0.988205891 \times (3/8)^2 + \ldots + 0.035868343 \times (3/8)^8 = 0.888913365$

b. For $0 < z \leq 1$, $\Gamma(z) = \Gamma(1 - x)$—use reflection formula.

$$\Gamma(z) = \Gamma(1-x) = \frac{\pi x}{\Gamma(1+x)\sin(\pi x)}$$

Example: $\Gamma(5/8) = \Gamma(1 - 3/8) = \pi(3/8)/[\Gamma(1 + 3/8)\sin(3\pi/8)]$
$= \pi(3/8)/[0.888913365 \sin(3\pi/8)] = 1.434519178$

c. For $z > 2$ (i.e., $x > 1$), $\Gamma(z) = \Gamma(1 + x)$—use recurrence formula.

$$\Gamma(z) = \Gamma(1+x) = \left[\prod_{n=1}^{k}(z-n)\right]\Gamma(1+\alpha)$$

where $0 \leq \alpha < 1$, integer value $k = (x - \alpha)$, and $\Gamma(1 + \alpha)$ can be obtained from Equations i or ii.

Example: $\Gamma(5.2) = \Gamma(1 + 4.2) = (5.2 - 1) \times (5.2 - 2) \times (5.2 - 3) \times (5.2 - 4) \times \Gamma(1 + 0.2) = 35.4816 \times 0.918168911 = 32.578102$, where $z = 5.2$, $x = 4.2$, $\alpha = 0.2$, $k = (4.2 - 0.2) = 4$

- Evaluation of gamma function using spreadsheets

  Microsoft Excel—EXP(GAMMALN(z))

  EXP(GAMMALN(11/8)) = 0.888913569,
  EXP(GAMMALN(5/8)) = 1.434518848,

  EXP(GAMMALN(5.2)) = 32.57809604.

  Quattro Pro—@EXP(@GAMMALN(X))

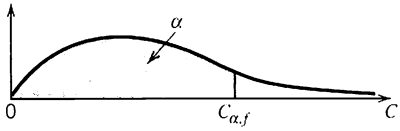
  @EXP(@GAMMALN(11/8)) = 0.88891336,
  @EXP(@GAMMALN(5/8)) = 1.43451904,

  @EXP(@GAMMALN(5.2)) = 32.57809603.

Appendix 3

Table of the CDF of the Chi-Square Distribution with f Degrees of Freedom

$$P(C \le c_{\alpha,f}) = \int_0^{c_{\alpha,f}} \frac{1}{2^{f/2}\Gamma(f/2)} s^{(f/2)-1} e^{-s/2} ds$$



f \ α	0.001	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995	0.999
1	1.57e-06	3.93e-05	1.57e-04	9.82e-04	3.93e-03	1.58e-02	2.706	3.841	5.024	6.635	7.879	10.83
2	2.00e-03	1.00e-02	0.0201	0.0506	0.1026	0.2107	4.605	5.991	7.378	9.210	10.60	13.82
3	2.43e-02	7.17e-02	0.1148	0.2158	0.3518	0.5844	6.251	7.815	9.348	11.34	12.84	16.27
4	9.08e-02	0.2070	0.2971	0.4844	0.7107	1.064	7.779	9.488	11.14	13.28	14.86	18.47
5	0.2102	0.4118	0.5543	0.8312	1.145	1.610	9.236	11.07	12.83	15.09	16.75	20.51
6	0.3810	0.6757	0.8721	1.237	1.635	2.204	10.64	12.59	14.45	16.81	18.55	22.46
7	0.5985	0.9893	1.239	1.690	2.167	2.833	12.02	14.07	16.01	18.48	20.28	24.32
8	0.8571	1.344	1.647	2.180	2.733	3.490	13.36	15.51	17.53	20.09	21.95	26.12
9	1.152	1.735	2.088	2.700	3.325	4.168	14.68	16.92	19.02	21.67	23.59	27.88
10	1.479	2.156	2.558	3.247	3.940	4.865	15.99	18.31	20.48	23.21	25.19	29.59
11	1.834	2.603	3.053	3.816	4.575	5.578	17.28	19.68	21.92	24.73	26.76	31.26
12	2.214	3.074	3.571	4.404	5.226	6.304	18.55	21.03	23.34	26.22	28.30	32.91
13	2.617	3.565	4.107	5.009	5.892	7.041	19.81	22.36	24.74	27.69	29.82	34.53
14	3.041	4.075	4.660	5.629	6.571	7.790	21.06	23.68	26.12	29.14	31.32	36.12
15	3.483	4.601	5.229	6.262	7.261	8.547	22.31	25.00	27.49	30.58	32.80	37.70
16	3.942	5.142	5.812	6.908	7.962	9.312	23.54	26.30	28.85	32.00	34.27	39.25
17	4.416	5.697	6.408	7.564	8.672	10.09	24.77	27.59	30.19	33.41	35.72	40.79
18	4.905	6.265	7.015	8.231	9.390	10.86	25.99	28.87	31.53	34.81	37.16	42.31
19	5.407	6.844	7.633	8.907	10.12	11.65	27.20	30.14	32.85	36.19	38.58	43.82
20	5.921	7.434	8.260	9.591	10.85	12.44	28.41	31.41	34.17	37.57	40.00	45.31

f \ α	0.001	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995	0.999
21	6.447	8.034	8.897	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40	46.80
22	6.983	8.643	9.542	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80	48.27
23	7.529	9.260	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18	49.73
24	8.085	9.886	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56	51.18
25	8.649	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93	52.62
26	9.222	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29	54.05
27	9.803	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.65	55.48
28	10.39	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99	56.89
29	10.99	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34	58.30
30	11.59	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67	59.70
40	17.92	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77	73.40
45	21.25	24.31	25.90	28.37	30.61	33.35	57.51	61.66	65.41	69.96	73.17	80.08
50	24.67	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49	86.66
55	28.17	31.73	33.57	36.40	38.96	42.06	68.80	73.31	77.38	82.29	85.75	93.17
60	31.74	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95	99.61
70	39.04	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.43	104.21	112.32
80	46.52	51.17	53.54	57.15	60.39	64.28	96.58	101.88	106.63	112.33	116.32	124.84
90	54.16	59.20	61.75	65.65	69.13	73.29	107.57	113.15	118.14	124.12	128.30	137.21
100	61.92	67.33	70.06	74.22	77.93	82.36	118.50	124.34	129.56	135.81	140.17	149.45

Appendix 4

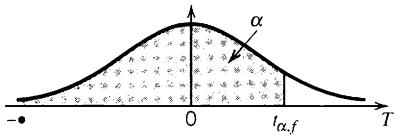
Values of D_n^{α} for the Kolmogorov–Smirnov Test

n	$D_n^{0.2}$	$D_n^{0.15}$	$D_n^{0.1}$	$D_n^{0.05}$	$D_n^{0.01}$
5	0.446	0.474	0.510	0.563	0.669
6	0.410	0.436	0.470	0.521	0.618
7	0.381	0.405	0.438	0.486	0.577
8	0.358	0.381	0.411	0.457	0.543
9	0.339	0.360	0.388	0.432	0.514
10	0.322	0.342	0.368	0.409	0.486
11	0.307	0.326	0.352	0.391	0.468
12	0.295	0.313	0.338	0.375	0.450
13	0.284	0.302	0.325	0.361	0.433
14	0.274	0.292	0.314	0.349	0.418
15	0.266	0.283	0.304	0.338	0.404
20	0.231	0.246	0.264	0.294	0.352
25	0.21	0.22	0.24	0.264	0.32
30	0.19	0.20	0.22	0.242	0.29
35	0.18	0.19	0.21	0.23	0.27
40	0.17	0.18	0.19	0.21	0.25
45	0.16	0.17	0.18	0.20	0.24
50	0.15	0.16	0.17	0.19	0.23
>50	$\frac{1.07}{\sqrt{n}}$	$\frac{1.14}{\sqrt{n}}$	$\frac{1.22}{\sqrt{n}}$	$\frac{1.36}{\sqrt{n}}$	$\frac{1.63}{\sqrt{n}}$

Appendix 5

Table of the CDF of Student's *t*-Distribution

$$P(T \le t_{\alpha,f}) = \int_{-\infty}^{t_{\alpha,f}} \frac{\Gamma[(f+1)]/2}{\sqrt{\pi f}\,\Gamma(f/2)} \left(1 + \frac{s^2}{f}\right)^{-(f+1)/2} ds$$



f \ α	0.600	0.750	0.800	0.900	0.950	0.975	0.990	0.995	0.999
1	0.325	1.000	1.376	3.078	6.314	12.706	31.821	63.656	318.289
2	0.289	0.816	1.061	1.886	2.920	4.303	6.965	9.925	22.328
3	0.277	0.765	0.978	1.638	2.353	3.182	4.541	5.841	10.214
4	0.271	0.741	0.941	1.533	2.132	2.776	3.747	4.604	7.173
5	0.267	0.727	0.920	1.476	2.015	2.571	3.365	4.032	5.894
6	0.265	0.718	0.906	1.440	1.943	2.447	3.143	3.707	5.208
7	0.263	0.711	0.896	1.415	1.895	2.365	2.998	3.499	4.785
8	0.262	0.706	0.889	1.397	1.860	2.306	2.896	3.355	4.501
9	0.261	0.703	0.883	1.383	1.833	2.262	2.821	3.250	4.297
10	0.260	0.700	0.879	1.372	1.812	2.228	2.764	3.169	4.144
11	0.260	0.697	0.876	1.363	1.796	2.201	2.718	3.106	4.025
12	0.259	0.695	0.873	1.356	1.782	2.179	2.681	3.055	3.930
13	0.259	0.694	0.870	1.350	1.771	2.160	2.650	3.012	3.852
14	0.258	0.692	0.868	1.345	1.761	2.145	2.624	2.977	3.787
15	0.258	0.691	0.866	1.341	1.753	2.131	2.602	2.947	3.733
16	0.258	0.690	0.865	1.337	1.746	2.120	2.583	2.921	3.686
17	0.257	0.689	0.863	1.333	1.740	2.110	2.567	2.898	3.646
18	0.257	0.688	0.862	1.330	1.734	2.101	2.552	2.878	3.610
19	0.257	0.688	0.861	1.328	1.729	2.093	2.539	2.861	3.579
20	0.257	0.687	0.860	1.325	1.725	2.086	2.528	2.845	3.552

f \ α	0.600	0.750	0.800	0.900	0.950	0.975	0.990	0.995	0.999
21	0.257	0.686	0.859	1.323	1.721	2.080	2.518	2.831	3.527
22	0.256	0.686	0.858	1.321	1.717	2.074	2.508	2.819	3.505
23	0.256	0.685	0.858	1.319	1.714	2.069	2.500	2.807	3.485
24	0.256	0.685	0.857	1.318	1.711	2.064	2.492	2.797	3.467
25	0.256	0.684	0.856	1.316	1.708	2.060	2.485	2.787	3.450
26	0.256	0.684	0.856	1.315	1.706	2.056	2.479	2.779	3.435
27	0.256	0.684	0.855	1.314	1.703	2.052	2.473	2.771	3.421
28	0.256	0.683	0.855	1.313	1.701	2.048	2.467	2.763	3.408
29	0.256	0.683	0.854	1.311	1.699	2.045	2.462	2.756	3.396
30	0.256	0.683	0.854	1.310	1.697	2.042	2.457	2.750	3.385
40	0.255	0.681	0.851	1.303	1.684	2.021	2.423	2.704	3.307
50	0.255	0.679	0.849	1.299	1.676	2.009	2.403	2.678	3.261
60	0.254	0.679	0.848	1.296	1.671	2.000	2.390	2.660	3.232
100	0.254	0.677	0.845	1.290	1.660	1.984	2.364	2.626	3.174
120	0.254	0.677	0.845	1.289	1.658	1.980	2.358	2.617	3.160
150	0.254	0.676	0.844	1.287	1.655	1.976	2.351	2.609	3.145
200	0.254	0.676	0.843	1.286	1.653	1.972	2.345	2.601	3.131
∞	0.253	0.674	0.842	1.282	1.645	1.960	2.326	2.576	3.090

Appendix 6

Gram–Schmidt Orthogonalization

Consider a matrix $\mathbf{R}_0$, with row vectors $\mathbf{r}_{01}$, $\mathbf{r}_{02}$, ..., $\mathbf{r}_{0n}$. This has to be transformed to a matrix $\mathbf{R}$, whose row vectors $\mathbf{r}_1$, $\mathbf{r}_2$, ..., $\mathbf{r}_n$ are orthogonal to each other, with the nth row the same as in matrix $\mathbf{R}_0$, that is, $\mathbf{r}_n = \mathbf{r}_{0n}$.

The Gram–Schmidt (G–S) method to achieve this may be written as follows. The nth row vector of matrix $\mathbf{R}$ is simply $\mathbf{r}_n = \mathbf{r}_{0n}$. The other rows of matrix $\mathbf{R}$ are computed [going from $(n-1)$th row to the first row, in that order] using the formula

$$\mathbf{r}_k = \mathbf{r}_{0k} - \sum_{j=k+1}^{n} \frac{\mathbf{r}_j \mathbf{r}_{0k}^t}{\mathbf{r}_j \mathbf{r}_j^t} \mathbf{r}_j \tag{A6.1}$$

where the superscript t implies the transpose of the row vector. Note that the rows of $\mathbf{R}$ have to be computed in the reverse order, from n to 1.

If the rows of $\mathbf{R}$ are required to be orthonormal (i.e., the rows are orthogonal to each other, and each row vector is of unit length), then each row should be normalized separately at the end.

Example (Application of the G–S Method to Structural Reliability)

Consider Example 8.1 in Section 8.2. The direction cosines of the β-vector are $\alpha_1 = 0.867$, $\alpha_2 = 0.498$. The coordinate system (Y_1, Y_2) has to be transformed to another orthogonal coordinate system (Y'_1, Y'_2) such that Y'_2 is along the β-vector. To do this, a transformation matrix $\mathbf{R}$ needs to be constructed such that $\mathbf{Y}' = \mathbf{RY}$.

First, a matrix $\mathbf{R}_0$ is selected as

$$\mathbf{R}_0 = \begin{bmatrix} 1 & 0 \\ 0.867 & 0.498 \end{bmatrix}$$

The rows of the transformed matrix $\mathbf{R}$ are obtained using the G–S method as follows. The second row is simply $\mathbf{r}_2 = [0.867\ 0.498]$. To compute the first row, use Equation A6.1. In this formula, $k = 1$ and $n = 2$. Therefore,

$$\mathbf{r}_1 = \{1 \quad 0\} - \frac{\{0.867 \quad 0.498\}\begin{Bmatrix}1\\0\end{Bmatrix}}{\{0.867 \quad 0.498\}\begin{Bmatrix}0.867\\0.498\end{Bmatrix}}\{0.867 \quad 0.498\}$$

$$= \{0.248 \quad -0.432\}.$$

Normalizing the elements of $\mathbf{r}_1$ to produce a unit vector, $\mathbf{r}_1 = \{0.498 \; -0.867\}$. Thus, the matrix $\mathbf{R}$ is obtained as

$$\mathbf{R} = \begin{bmatrix}0.498 & -0.867\\0.867 & 0.498\end{bmatrix}$$

This is identical to the result obtained with the special equation for two variables in Equation 8.4.

CONVERSION FACTORS

		Customary to SI
inches (in.)	meters (m)	0.0254
inches (in.)	centimeters (cm)	2.54
inches (in.)	millimeters (mm)	25.4
feet (ft)	meters (m)	0.305
yards (yd)	meters (m)	0.914
miles (miles)	kilometers (km)	1.609
degrees (°)	radians (rad)	0.0174
acres (acre)	hectares (ha)	0.405
acre-feet (acre-ft)	cubic meters (m^3)	1233
gallons (gal)	cubic meters (m^3)	3.79×10^{-3}
gallons (gal)	liters (l)	3.79
pounds (lb)	kilograms (kg)	0.4536
tons (ton, 2000 lb)	kilograms (kg)	907.2
pound force (lbf)	newtons (N)	4.448
pounds per sq in. (psi)	newtons per sq m (N/m^2)	6895
pounds per sq ft (psf)	newtons per sq m (N/m^2)	47.88
foot-pounds (ft-lb)	joules (J)	1.356
horsepowers (hp)	watts (W)	746
British thermal units (BTU)	joules (J)	1055
British thermal units (BTU)	kilowatt-hours (kwh)	2.93×10^{-4}

DEFINITIONS

newton—force that will give a 1-kg mass an acceleration of 1 m/sec^2
joule—work done by a force of 1 N over a displacement of 1 m
1 newton per sq m (N/m^2) = 1 pascal
1 kilogram force (kgf) = 9.807 N
1 gravity acceleration (g) = 9.807 m/sec^2
1 are (a) = 100 m^2
1 hectare (ha) = 10,000 m^2
1 kip (kip) = 1000 lb

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